CUK - H00565 - 25 - P17589

BULLETIN

of the

CALCUTTA MATHEMATICAL SOCIETY

290/25

VOLUME 39

P17589

1947

510·5
DCCA

BOARD OF EDITORS

N. M. BASU, D.SC. F. W. LEVI, DR. PHIL. NAT.
N. R. SEN, D.SC., PH.D. M. R. SIDDIQI, M.A., PH.D.
R. N. SEN, PH.D. C. V. H. RAO, M.A.
S. GUPTA, M.SC. (*Editorial Secretary*)

PUBLISHED BY THE CALCUTTA MATHEMATICAL SOCIETY
92, UPPER CIRCULAR ROAD, CALCUTTA
1947

CALCUTTA MATHEMATICAL SOCIETY

OFFICERS FOR 1947

President

Prof. N. M. Basu, D.Sc.

Vice-Presidents

Prof. A. C. Banerjee, M.A.
Dr. R. N. Sen, M.A., Ph.D.
Prof. N. R. Sen, D.Sc., Ph.D.
Dr. S. S. Pillai, D.Sc.
Proof. M. R. Siddiqi, M.A., Ph.D.

Treasurer

Mr. S. C. Ghosh, M.A.

Secretary

Mr. U. R. Burman, M.Sc.

Editorial Secretary

Mr. S. Gupta, M.Sc.

Assistant Secretary

Mr. P. K. Ghosh, M.Sc.

PRINTED IN INDIA

PRINTED BY NISHITCHANDRA SEN, SUPERINTENDENT (OFFG.), CALCUTTA UNIVERSITY PRESS,
48, HAZRA ROAD, BALLYGUNGE, CALCUTTA

19

CONTENTS

PAGE

Banerjee, D. P. On the divisors of numbers 57

Chakrabarti, M. C. On a special case of the distribution law of the mean square successive difference 15

——— Remainders in quadrature formulae 119

——— On the inadequacy of measuring the peakedness of a distribution curve by the stardardised fourth moment 154

Chakrabarty, S. K. Generation of mesons and its dependence on meson spin ... 166

Das Gupta, P. N. and **Upadhyay, S. D.** On a generalised continued fraction ... 65

Fuchs, Ladislas. Further generalisation of notion of relatively prime ideals ... 143

Ghosh, N. L. A note on the equilibrium of fluid matter in a steady differential rotation 181

Ghosh, P. K. On $(C, 1)$-convergent integrals and their application to mathematical physics 19

Ghosh, S. On the flexure of an isotropic elastic cylinder 1

——— On a new function-theoretic method of solving torsion problem of some boundaries 107

Hu, Sze-tsen. Homotopy groups of some mapping spaces of spheres 127

Karmarkar, K. R. An important particular case of the problem of equivalence ... 30

Kerawala, S. M. The asymptotic number of three-deep Latin rectangles ... 71

——— Asymptotic solution of the "Problème des Ménages." 82

——— Poncelet porism in two circles 85

Lahiri, D. B. On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$—II ... 33

Moessner, Alfred. Einige diophantische Probleme und Resultat 78

Nandi, H. K. A note on student' t for paired samples 61

Nasir, Abdur Rahman. Properties of three numbers in A-G. P. 79

Ray, M. On a type of non-stationary turbulent wake 189

Sen, Bibhutibhusan. Direct determination of stresses in thin elastic plates having cavities of different shapes 113

Sen Gupta, B. K. Propositions on polar reciprocals 59

Sengupta, N. D. On an exact solution of Dirac electron in the field of electromagnetic radiation 147

Upadhyay, S. D. and **Das Gupta, P. N.** On a generalised continued fraction ... 65

Zahorski, Z. On zeros of quasi-analytic (B) functions 157

Annual Report 58

Balance Sheet 56

CALCUTTA UNIVERSITY LIBRARY

ON THE FLEXURE OF AN ISOTROPIC ELASTIC CYLINDER

By

S. Ghosh

(*Received January 21, 1947*)

1. Among the new methods which have recently been proposed for the solution of the problem of torsion of an isotropic elastic cylinder, that of Muschelisvili (1929) stands out prominently because of its remarkable simplicity and elegance. His method consists in making a conformal representation of the cross-section of the cylinder on the unit circle, and then reducing the torsion problem to the determination of a function which is analytic within this circle, and whose imaginary part takes prescribed values on its boundary. The required function is then determined with the help of Schwarz's formula.

No attempt has, as yet, been made to apply this function-theoretic method to the solution of the general problem of flexure of cylindrical beams, and the reason is not far to seek. The boundary condition in the flexure problem, when it is formulated as a Dirichlet problem, is not so simple as that in the torsion problem. The boundary value of the harmonic function sought for in a torsion problem is very simple, viz., $\frac{1}{2}(x^2+y^2)$, whereas the boundary value of the corresponding function in the flexure problem involves a curvilinear integral. It is because of this difficulty that the flexure problem has been, for a long time after its first formulation by Saint-Venant, considered as a Neumann problem till its treatment by Timoshenko (1913, 1922) as a boundary value problem of the first kind. Griffith and Taylor (1921), Nemenyi (1921) and Sokolnikoff (1946, p. 243) have recently formulated the flexure problem purely as a Dirichlet problem.

In this paper, the function-theoretic method has been applied to the general problem of flexure. For this purpose, the problem has been sub-divided into three simpler Dirichlet problems with boundary conditions (4.1), (4.2) and (4.3). For such a sub-division to be possible, it is essential that the origin should be taken at the centroid of a cross-section. Regarding the choice of axes in the plane of the cross-section, it is to be mentioned that there is neither any advantage nor any disadvantage in departing from the usual procedure of taking them as the principal axes of the cross-section at its centroid. If the mapping function necessary for the conformal representation of a cross-section on the unit circle be known when the equation of the boundary of the cross-section is referred to any convenient system of axes at the centroid, the corresponding mapping function when the boundary is referred to any other set of axes at the centroid, is obtained by multiplying the previous mapping function by a constant.

The orientation of the load-line in the terminal cross-section has been taken arbitrarily, and not necessarily passing through the centroid of the cross-section.

Stevenson (1938) has shown that this procedure leads to the identification of a definite point on the cross-section, called the centre of flexure. Its importance lies in the fact that the solution of the general flexure problem can be obtained by superposing the solutions, (*i*) of a problem of pure flexure (unassociated with torsion) with the load-line passing through the centre of flexure, and (*ii*) of a problem of pure torsion.

The first of the three simpler Dirichlet problems into which the general flexure problem has been sub-divided is the problem of torsion and has already been treated by the function-theoretic method by Muschelisvili (1929). For the application of the method to the two other problems, the boundary conditions (4.2) and (4.3) for ψ_1 and ψ_2 have been combined into a single boundary condition which gives the boundary value of $\psi_2+i\psi_1$. This boundary value has then been divided into parts, as shown in (11.1). The analytic functions f_1 and f_2 with ψ_1 and ψ_2 as their imaginary parts, have then been expressed in terms of four analytic functions $f_{11}, f_{12}, f_{21}, f_{22}$ whose imaginary parts take prescribed values on the boundary.

Schwarz's formula (Villat, 1938, p. 58)

$$f(z) = i\beta_0+\frac{1}{2\pi}\int_0^{2\pi} g(\epsilon)\,\frac{1+z\exp(-i\epsilon)}{1-z\exp(-i\epsilon)}\,d\epsilon, \tag{1.1}$$

giving the value of a function $f(z)$, analytic within the unit circle, whose real part takes the value $g(\epsilon)$ at a point of the circumference whose angular distance from Ox is ϵ, can be replaced by

$$f(z) = i\beta_0-\frac{1}{2\pi}\int_0^{2\pi} g(\epsilon)d\epsilon+\frac{1}{\pi i}\int_\gamma \frac{\theta(t)dt}{t-z}, \tag{1.2}$$

by putting $t = \exp(i\epsilon)$, $g(\epsilon) = \theta(t)$, and taking the last integral over the circumference of the unit circle. In the torsion-flexure problem any additive constant in the function sought for, can be omitted. Therefore the analytic function $f(z)/i$ whose real part takes the value $\theta(t)$ on the circumference of the unit circle is given by

$$f(z) = \frac{1}{\pi}\int_\gamma \frac{\theta(t)dt}{t-z}. \tag{1.3}$$

Schwarz's formula, in this modified form, has been used for the determination of the four flexure functions $f_{11}, f_{12}, f_{21}, f_{22}$.

2. Let the cross-section at one end of a cylinder of length l be fixed. Let a pair of rectangular axes, lying in the plane of this cross-section and meeting at its centroid, be taken as the axes of x and y, and let the central line of the cylinder be taken as the axis of z. Assuming that

$$X_x = Y_y = X_y = 0, \quad Z_z = -(l-z)(k_1x+k_2y), \tag{2.1}$$

we find from the stress equations of equilibrium that X_z, Y_z are independent of z, and that

$$\frac{\partial}{\partial x}\left[X_z+\frac{1}{2}k_1\left(x^2-\frac{\sigma}{1+\sigma}y^2\right)\right]+\frac{\partial}{\partial y}\left[Y_z+\frac{1}{2}k_2\left(y^2-\frac{\sigma}{1+\sigma}x^2\right)\right]=0. \tag{2,2}$$

Four of the stress compatibility equations (Love, p. 135) are satisfied, and the remaining two reduce to

$$\nabla_1^2\left[X_z+\frac{1}{2}k_1\left(x^2-\frac{\sigma}{1+\sigma}y^2\right)\right]=0,\qquad \nabla_1^2\left[Y_z+\frac{1}{2}k_2\left(y^2-\frac{\sigma}{1+\sigma}x^2\right)\right]=0. \tag{2.3}$$

From (2.2) and (2.3), we see that

$$X_z+\frac{1}{2}k_1\left(x^2-\frac{\sigma}{1+\sigma}y^2\right)=\frac{\partial\psi'}{\partial y},\quad Y_z+\frac{1}{2}k_2\left(y^2-\frac{\sigma}{1+\sigma}x^2\right)=-\frac{\partial\psi'}{\partial x}, \tag{2.4}$$

where $\psi'(x, y)$ satisfies the equation

$$\nabla_1^2\psi' = \text{const} = k_0. \tag{2.5}$$

Putting

$$\psi' = \psi+\tfrac{1}{4}k_0(x^2+y^2), \tag{2.6}$$

we find that

$$\nabla_1^2\psi = 0. \tag{2.7}$$

Since the surface of the cylinder is free from traction, we have

$$X_z\frac{dy}{ds}-Y_z\frac{dx}{ds}=0,$$

on the contour C of a cross-section. Substituting from (2.4) and (2.6), we get

$$\psi=-\tfrac{1}{4}k_0(x^2+y^2)+\frac{1}{2}\int_A^P\left[k_1\left(x^2-\frac{\sigma}{1+\sigma}y^2\right)dy-k_2\left(y^2-\frac{\sigma}{1+\sigma}x^2\right)dx\right]+\text{const}, \tag{2.8}$$

on C, the curvilinear integral being taken along C from a fixed point A to a variable point P. Now

$$\int_C\left[k_1\left(x^2-\frac{\sigma}{1+\sigma}y^2\right)dy-k_2\left(y^2-\frac{\sigma}{1+\sigma}x^2\right)dx\right]=2\iint(k_1x+k_2y)\,dx\,dy,$$

where the double integral is taken over the area of the cross-section. Since the centroid of the cross-section is the origin, this double integral vanishes. Consequently ψ, as given by (2.8), is a single-valued function of x, y.

3. Let X, Y, Z, L, M, N be the force and the couple components of the resultant tractions across a cross-section $z = z_0$, referred to parallel axes through its centroid. It can very easily be shown that

$$X = k_1I_{11}+k_2I_{12},\quad Y = k_1I_{12}+k_2I_{22},\quad Z = 0, \tag{3.1}$$

$$L = -(l-z_0)(k_1I_{12}+k_2I_{22}),\quad M = (l-z_0)(k_1I_{11}+k_2I_{12}), \tag{3.2}$$

$$N = \iint(xY_z-yX_z)\,dx\,dy, \tag{3.3}$$

where

$$I_{11} = \iint x^2\,dx\,dy,\quad I_{12} = \iint xy\,dx\,dy,\quad I_{22} = \iint y^2\,dx\,dy, \tag{3.4}$$

the integrals in (3.3) and (3.4) being taken over the area of the cross section. N is known when ψ is known.

As $LX+MY+NZ=0$, the tractions across a cross-section $z=z_0$ reduce to a single force $(P_1, P_2, 0)$, say, whose line of action is given by

$$(L-yZ+z'Y)/X = (M-z'X+xZ)/Y, \quad N-xY+yX=0,$$

where $z' = z-z_0$ is the z coordinate of a point referred to the centroid of the cross-section as origin. Putting

$$k_1I_{11}+k_2I_{12} = P_1, \quad k_1I_{12}+k_2I_{22} = P_2, \tag{3.5}$$

these equations reduce to

$$xP_2-yP_1 = N, \quad z' = l-z_0 \text{ or } z = l. \tag{3.6}$$

Thus the stress-system we have assumed solves the problem of flexure of a cylinder by a terminal load acting in the plane $z=l$.

Solving the equations (3.5) for k_1, k_2, we get

$$k_1 = (P_1I_{22}-P_2I_{12})/(I_{11}I_{22}-I_{12}^2), \quad k_2 = (P_2I_{11}-P_1I_{12})/(I_{11}I_{22}-I_{12}^2). \tag{3.7}$$

If the axes of x, y are the principal axes at the centroid, $I_{12}=0$, and we have

$$k_1 = P_1/I_{11}, \quad k_2 = P_2/I_{22}. \tag{3.8}$$

4. Introducing plane harmonic functions ψ_0, ψ_1, ψ_2 satisfying the boundary conditions,

$$\psi_0 = \tfrac{1}{2}(x^2+y^2)+\text{const.}, \tag{4.1}$$

$$\psi_1 = \int_A^P [(1+\sigma)x^2-\sigma y^2]dy+\text{const.}, \tag{4.2}$$

$$\psi_2 = -\int_A^P [(1+\sigma)y^2-\sigma x^2]dx+\text{const.}, \tag{4.3}$$

we see that we can write

$$\psi = -\tfrac{1}{2}k_0\psi_0+\tfrac{1}{2}(k_1\psi_1+k_2\psi_2)/(1+\sigma). \tag{4.4}$$

It can be shown, as before, that (4.1), (4.2) and (4.3) lead to single-valued functions ψ_0, ψ_1, ψ_2.

5. In terms of these new functions,

$$X_z = -\frac{1}{2}k_0\left(\frac{\partial\psi_0}{\partial y}-y\right)+\frac{k_1}{2(1+\sigma)}\left[\frac{\partial\psi_1}{\partial y}-(1+\sigma)x^2+\sigma y^2\right]+\frac{k_2}{2(1+\sigma)}\frac{\partial\psi_2}{\partial y}, \tag{5.1}$$

$$Y_z = \frac{1}{2}k_0\left(\frac{\partial\psi_0}{\partial x}-x\right)-\frac{k_1}{2(1+\sigma)}\frac{\partial\psi_1}{\partial x}-\frac{k_2}{2(1+\sigma)}\left[\frac{\partial\psi_2}{\partial x}+(1+\sigma)y^2-\sigma x^2\right]. \tag{5.2}$$

6. The displacements can be calculated in the same way as in Love (p. 384), and we find, using the relation $2\mu = E/(1+\sigma)$,

$$u = \frac{k_0}{2\mu}yz+\frac{k_1}{2E}\left[\sigma(l-z)(x^2-y^2)+lz^2-\tfrac{1}{3}z^3\right]+\frac{\sigma k_2}{E}(l-z)xy, \tag{6.1}$$

$$v = -\frac{k_0}{2\mu}xz+\frac{\sigma k_1}{E}(l-z)xy+\frac{k_2}{2E}\left[\sigma(l-z)(y^2-x^2)+lz^2-\tfrac{1}{3}z^3\right], \tag{6.2}$$

$$w = -\frac{k_0}{2\mu}\phi_0 - \frac{k_1}{E}\left[x(lz-\tfrac{1}{2}z^2)-\phi_1+\tfrac{1}{6}(2+\sigma)x^3-\tfrac{1}{2}\sigma xy^2\right]$$

$$-\frac{k_2}{E}\left[y(lz-\tfrac{1}{2}z^2)-\phi_2+\tfrac{1}{6}(2+\sigma)y^3-\tfrac{1}{2}\sigma x^2y\right], \qquad (6.3)$$

where ϕ_0, ϕ_1, ϕ_2 are plane harmonic functions whose conjugates are ψ_0, ψ_1, ψ_2 respectively.

7. The boundary condition (4.1) and the parts

$$-\frac{1}{2}k_0\left(\frac{\partial\psi_0}{\partial y}-y\right), \quad \frac{1}{2}k_0\left(\frac{\partial\psi_0}{\partial x}-x\right)$$

of the stresses X_z, Y_z given by (5.1) and (5.2) respectively, or the parts

$$\frac{k_0}{2\mu}yz, \quad -\frac{k_0}{2\mu}xz, \quad -\frac{k_0}{2\mu}\phi_0$$

of the displacements u, v, w given by (6.1), (6.2) and (6.3) respectively, show that ϕ_0 is the torsion function for the boundary C, ψ_0 its conjugate harmonic and $-\frac{1}{2}k_0/\mu$ the twist per unit length of the cylinder. Therefore, unless $k_0 = 0$, the flexure of the cylinder is associated with a torsion. We will call ψ_1, ψ_2 the flexure functions for the boundary C. We observe that when the coordinate axes are the principal axes at the centroid of a cross-section, the functions ϕ_1, ϕ_2 whose conjugate harmonics are ψ_1, ψ_2, are not the classical Saint-Venant flexure functions for the boundary. In fact, a comparison of the formulæ (5.1), (5.2) and (3.3) or of the formulæ (6.1), (6.2), (6.3) and (3.3) with the corresponding formulæ given by Love (p. 332 and p. 334) shows that the classical Saint-Venant flexure functions for the boundary are

$$\chi_1 = -\phi_1+\tfrac{1}{6}(2+\sigma)(x^3-3xy^2), \quad \chi_2 = -\phi_2-\tfrac{1}{6}(2+\sigma)(3x^2y-y^3). \qquad (7.1)$$

The functions ψ_1, ψ_2 were introduced by Griffith and Taylor (1921) and Nemenyi (1921), using the principal axes at the centroid of a cross-section as the coordinate axes.

From the equations of the boundary, we can write

$$x^2 = f(y) \quad \text{and} \quad y^2 = g(x). \qquad (7.2)$$

If we introduce these relations in (4.2) and (4.3), they become

$$\psi_1 = (1+\sigma)\int f(y)dy - \tfrac{1}{3}\sigma y^3 + \text{const}, \qquad (7.3)$$

$$\psi_2 = -(1+\sigma)\int g(x)dx + \tfrac{1}{3}\sigma x^3 + \text{const}, \qquad (7.4)$$

so that the Dirichlet problems for the determination of ψ_1, ψ_2 can, at least theoretically, be solved. In case of many boundaries, $f(y)$ and $g(x)$ are such simple functions of y and x respectively, that the solution of the Dirichlet problems presents no difficulty.

If, instead of using the function ψ, we introduce the function ψ'' defined by

$$\psi'' = \psi + \tfrac{1}{4}k_0(x^2+y^2) - \frac{P_1}{2I_{11}}\int f(y)dy + \frac{P_2}{2I_{22}}\int g(x)dx + \frac{1}{6}\cdot\frac{\sigma}{1+\sigma}\left[\frac{P_1}{I_{11}}y^3 - \frac{P_2}{I_{22}}x^3\right], \quad (7.5)$$

we see that ψ'' satisfies the equation

$$\nabla_1{}^2\psi'' = k_0 - \frac{P_1}{2I_{22}}\frac{df}{dy} + \frac{P_2}{2I_{11}}\frac{dg}{dx} - \frac{\sigma}{1+\sigma}\left[\frac{P_1}{I_{22}}y - \frac{P_2}{I_{11}}x\right] \quad (7.6)$$

within the cross-section; and

$$\psi'' = \text{const.}, \quad (7.7)$$

on the boundary. The constant in (7.7) can evidently be taken to be zero. The function ψ'' was first introduced by Timoshenko (1913, 1922) in solving the flexure problem for a number of boundaries and in formulating the membrane analogy of flexure,

8. Going back to arbitrary rectangular axes at the centroid of a cross-section, and substituting for X_z, Y_z from (5.1) and (5.2) in (3.3), we get

$$N = -\tfrac{1}{2}k_0N_0 + \tfrac{1}{2}(k_1N_1 + k_2N_2)/(1+\sigma), \quad (8.1)$$

where

$$N_0 = \iint\left[x^2+y^2 - x\frac{\partial\psi_0}{\partial x} - y\frac{\partial\psi_0}{\partial y}\right]dxdy, \quad (8.2)$$

$$N_1 = \iint\left[(1+\sigma)x^2y - \sigma y^3 - x\frac{\partial\psi_1}{\partial x} - y\frac{\partial\psi_1}{\partial y}\right]dxdy, \quad (8.3)$$

$$N_2 = \iint\left[\sigma x^3 - (1+\sigma)xy^2 - x\frac{\partial\psi_2}{\partial x} - y\frac{\partial\psi_2}{\partial y}\right]dxdy, \quad (8.4)$$

the integrals being taken over the area of the cross-section.

The line of action of the resultant of tractions on the terminal cross-section $z = l$ of the cylinder is given by

$$xP_2 - yP_1 = -\tfrac{1}{2}k_0N_0 + \tfrac{1}{2}(k_1N_1 + k_2N_2)/(1+\sigma), \quad z = l. \quad (8.5)$$

If, as in Saint-Venant flexure, the load acts at the centroid, this straight line passes through the point $(0, 0, l)$, so that

$$-\tfrac{1}{2}k_0N_0 + \tfrac{1}{2}(k_1N_1 + k_2N_2)/(1+\sigma) = 0. \quad (8.6)$$

This equation determines k_0, and consequently the twist per unit length of the cylinder in the torsion associated with the flexure. If the load passes through the point (x_1, y_1, l), k_0 is determined from the first of the equations (8.5), in which we put $x = x_1$, $y = y_1$. The most important case arises when there is no associated torsion, *i.e.*, when $k_0 = 0$. From (8.5) and (8.7) it is seen that the line of action of the load is given by

$$P_2\left[x - \frac{I_{11}N_2 - I_{12}N_1}{2(1+\sigma)(I_{11}I_{22} - I_{12}^2)}\right] - P_1\left[y - \frac{I_{12}N_2 - I_{22}N_1}{2(1+\sigma)(I_{11}I_{22} - I_{12}^2)}\right] = 0, \quad z = l. \quad (8.7)$$

Thus, for a load of given direction (*i.e.*, for a given value of the ratio $P_1 : P_2$) there is a definite line on the terminal cross-section, parallel to the given direction, such that if the load acts along this line at any point on it, no torsion is associated with the flexure. This line will be called the line of flexure for the given direction. The equations (8.7) show that all lines of flexure pass through the point

$$x_f = \frac{I_{11}N_2 - I_{12}N_1}{2(1+\sigma)(I_{11}I_{22}-I_{12}^2)}, \quad y_f = \frac{I_{12}N_2 - I_{22}N_1}{2(1+\sigma)(I_{11}I_{22}-I_{12}^2)}, \quad z = l. \tag{8.8}$$

Therefore, if the load acts at the point (x_f, y_f, l) in the plane of the terminal cross-section in any direction in that plane, no torsion is associated with the flexure. This point is called the centre of flexure. The importance of the centre of flexure in the torsion-flexure theory is due to the fact that it enables us to obtain the solution of the flexure problem for the load $(P_1, P_2, 0)$ at the point (x_1, y_1, l) by superposition of the solution of the pure flexure problem by a load $(P_1, P_2, 0)$ acting at the centre of flexure and the solution of the torsion problem by a couple of moment

$$(x_1 - x_f)P_2 - (y_1 - y_f)P_1.$$

If the axis of x is an axis of symmetry of the cross-section, it is seen at once that $\psi_1(x, y)$ is an odd function of y so that the integrand in (8.8) is also odd in y. This gives $N_1 = 0$. Hence the centre of flexure lies on the axis of symmetry.

9. Let us now apply the function-theoretic method to obtain the solution of the flexure problem. Henceforth we will use z for $x+iy$. Let R be the simply-connected region in the z plane, which forms the cross-section of the beam, and let C be its boundary. Let $z = \omega(\zeta)$ give the conformal representation of this region on the unit circle $|\zeta| \leqq 1$, in the ζ-plane, and let γ be the circumference of this circle.

The area of the cross-section of the cylinder is given by

$$I_0 = \frac{1}{2}\iint_R \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right) dxdy = \frac{1}{2}\int_C (xdy - ydx) = -\frac{1}{2} i \int_C \bar{z}\, dz.$$

If $\bar{\omega}(\zeta)$ be the complex conjugate of $\omega(\zeta)$,

$$I_0 = -\frac{1}{2} i \int_\gamma \bar{\omega}(\bar{\zeta}) d\omega(\zeta) = -\frac{1}{2} i \int_\gamma \bar{\omega}(1/\zeta) d\omega(\zeta), \tag{9.1}$$

since $\bar{\zeta} = 1/\zeta$ on the unit circle.

The coordinates (I_1, I_2) of the centroid of the cross-section are given by

$$I_0 I_1 = \frac{1}{2}\iint_R \frac{\partial}{\partial x}(x^2+y^2)\, dxdy = \frac{1}{2}\int_C (x^2+y^2)\, dy,$$

$$I_0 I_2 = \frac{1}{2}\iint_R \frac{\partial}{\partial y}(x^2+y^2)\, dxdy = -\frac{1}{2}\int_C (x^2+y^2)\, dx.$$

If the origin be at the centroid, $I_1 = I_2 = 0$. If not, we have

$$I_0 z_0 = -\frac{1}{2} i \int_C z\bar{z}\, dz = -\frac{1}{2} i \int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta) d\omega(\zeta) \tag{9.2}$$

where $z_c = I_1 + iI_2$. If we now transfer the origin to the point $z = z_c$, the formula $z + z_c = \omega(\zeta)$ gives the conformal representation of the region R on the unit circle $|\zeta| \leqq 1$. We will hereafter take the mapping function as

$$z = \omega(\zeta), \tag{9.3}$$

with the origin of the z-plane at the centroid of the cross-section.

If the axes of x' and y' are obtained from the axes of x and y by turning them through an angle α about the origin, and if these new axes are taken as the axes of reference, then

$$z' = \omega(\zeta) \exp(-i\alpha) \tag{9.4}$$

maps the area of the cross-section on the unit circle $|\zeta| \leqq 1$. If

$$I'_{11} = \iint_R x'^2 dx'dy', \quad I'_{12} = \iint_R x'y'dx'dy', \quad I'_{22} = \iint_R y'^2 dx'dy', \tag{9.5}$$

then

$$\begin{aligned} I'_{11} + I'_{22} &= \frac{1}{4} \iint_R \left[\frac{\partial}{\partial x'}\left\{x'(x'^2+y'^2)\right\} + \frac{\partial}{\partial y'}\left\{y'(x'^2+y'^2)\right\}\right] dx'dy' \\ &= \frac{1}{4} \int_C (x'^2+y'^2)(x'dy'-y'dx') = -\frac{1}{4} i \int_C z'\bar{z}'^2 dz' \\ &= -\frac{1}{4} i \int_\gamma \omega(\zeta)[\bar{\omega}(1/\zeta)]^2 d\omega(\zeta), \end{aligned} \tag{9.6}$$

Also

$$I'_{11} - I'_{22} = \frac{1}{2} \int_C (x'^2+y'^2)(x'dy'+y'dx'),$$

$$\begin{aligned} I'_{12} &= \frac{1}{4} \iint_R \left[\frac{\partial}{\partial x'}\left\{y'(x'^2+y'^2)\right\} + \frac{\partial}{\partial y'}\left\{x'(x'^2+y'^2)\right\}\right] dx'dy' \\ &= \frac{1}{4} \int_C (x'^2+y'^2)(y'dy'-x'dx'), \end{aligned}$$

so that

$$I'_{11} - I'_{22} + 2iI'_{12} = -\frac{1}{2} i \int_C z'^2 \bar{z}' dz' = -\tfrac{1}{2} i \exp(-2i\alpha). \int_\gamma [\omega(\zeta)]^2 \bar{\omega}(1/\zeta) d\omega(\zeta). \tag{9.7}$$

If we choose α in such a manner that the right-hand side of (9.7) is real, then $I'_{12} = 0$, and the axes of x', y' are the principal axes of the cross-section at the centroid. With this value of α, the mapping function is given by (9.4). Consequently, the use of the principal axes at the centroid leads to no complication at all.

10. Henceforth we will assume that (9.3) gives the mapping function, the axes in the plane of the cross-section being any pair of rectangular axes at its centroid. The moments and products of inertia are then given by formulæ of types (9.6) and (9.7), when we put $\alpha = 0$ in them.

The solution of the torsion problem has been given by Muschelisvili (1929) in the form

$$f_0 = \phi_0 + i\psi_0 = \frac{1}{2\pi}\int_\gamma \frac{\omega(\sigma)\bar{\omega}(1/\sigma)}{\sigma-\zeta}\,d\sigma. \tag{10.1}$$

The torsional rigidity of the cylinder is μN_0. We have already proved in (9.6) that

$$\iint_R (x^2+y^2)\,dx\,dy = -\frac{1}{4}i\int_\gamma \omega(\zeta)[\bar{\omega}(1/\zeta)]^2 d\omega(\zeta).$$

Also

$$-\iint_R\left(x\frac{\partial\psi_0}{\partial x}+y\frac{\partial\psi_0}{\partial y}\right)dx\,dy = -\int_C \phi_0(x\,dx+y\,dy)$$

$$= \frac{1}{2}\int_C (x^2+y^2)\,d\phi_0 = \frac{1}{2}\mathcal{R}\int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)\,df_0(\zeta),$$

$\mathcal{R}$ denoting the real part of the integral. Hence

$$N_0 = -\frac{1}{4}i\int_\gamma \omega(\zeta)[\bar{\omega}(1/\zeta)]^2 d\omega(\zeta) + \frac{1}{2}\mathcal{R}\int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)\,df_0(\zeta). \tag{10.2}$$

11. For the determination of the flexure functions, we write the boundary values of ψ_1, ψ_2, as given by (4·2), (4.3), in the forms

$$\psi_1 = \frac{1}{4}\int_A^P [(1+2\sigma)(z^2+\bar{z}^2)+2z\bar{z}]\,dy + \text{const.},$$

$$\psi_2 = \frac{1}{4}\int_A^P [(1+2\sigma)(z^2+\bar{z}^2)-2z\bar{z}]\,dx + \text{const.},$$

whence we have on C,

$$\psi_2 + i\psi_1 = \tfrac{1}{12}(1+2\sigma)z^3 - \tfrac{1}{4}z\bar{z}^2 + \tfrac{1}{2}(1+\sigma)\int_A^P \bar{z}^2 dz + \text{const.} \tag{11.1}$$

Let f_{11}, f_{21} be analytic functions of z within C, whose imaginary parts take up values on C, which are respectively the imaginary and the real parts of $-\frac{1}{4}z\bar{z}^2$. Therefore f_{11}, f_{21} are analytic functions of ζ within γ, whose imaginary parts take up values

$$-\frac{1}{8i}z\bar{z}(\bar{z}-z) = -\frac{1}{8i}\omega(\zeta)\bar{\omega}(\bar{\zeta})[\bar{\omega}(\bar{\zeta})-\omega(\zeta)]$$

and

$$-\tfrac{1}{8}z\bar{z}(\bar{z}+z) = -\tfrac{1}{8}\omega(\zeta)\bar{\omega}(\bar{\zeta})[\bar{\omega}(\bar{\zeta})+\omega(\zeta)]$$

respectively on γ. Hence, by Schwarz's formula (1.3),

$$f_{11} = \frac{i}{8\pi}\int_\gamma \frac{\omega(\sigma)\bar{\omega}(1/\sigma)[\bar{\omega}(1/\sigma)-\omega(\sigma)]}{\sigma-\zeta}\,d\sigma,$$

$$f_{21} = -\frac{1}{8\pi}\int_\gamma \frac{\omega(\sigma)\bar{\omega}(1/\sigma)[\bar{\omega}(1/\sigma)+\omega(\sigma)]}{\sigma-\zeta}\,d\sigma.$$

Therefore

$$f_{11}+if_{21} = -\frac{i}{4\pi}\int_\gamma \frac{[\omega(\sigma)]^2\bar{\omega}(1/\sigma)d\sigma}{\sigma-\zeta}, \tag{11.2}$$

$$f_{11}-if_{21} = \frac{i}{4\pi}\int_\gamma \frac{\omega(\sigma)[\bar{\omega}(1/\sigma)]^2 d\sigma}{\sigma-\zeta}. \tag{11.3}$$

Again, let f_{12}, f_{22} be analytic functions of z within C, whose imaginary parts take values on C, which are respectively the imaginary and the real parts of

$$\frac{1}{2}\int^z \bar{z}^2 dz$$

taken along C. Hence f_{12}, f_{22} are analytic functions of ζ within γ whose imaginary parts take values on γ, which are respectively the imaginary and the real parts of

$$F(\zeta) = \frac{1}{2}\int^\zeta [\bar{\omega}(1/\zeta)]^2 d\omega(\zeta), \tag{11.4}$$

taken along γ, *i.e.*, whose imaginary parts take up values

$$\frac{1}{2i}[F(\zeta)-\bar{F}(\bar{\zeta})], \quad \frac{1}{2}[F(\zeta)+\bar{F}(\bar{\zeta})]$$

respectively, on γ. Hence, by Schwarz's formula (1.3),

$$f_{12} = -\frac{i}{2\pi}\int_\gamma \frac{F(\sigma)-\bar{F}(1/\sigma)}{\sigma-\zeta}\,d\sigma,$$

$$f_{22} = \frac{1}{2\pi}\int_\gamma \frac{F(\sigma)+\bar{F}(1/\sigma)}{\sigma-\zeta}\,d\sigma.$$

Therefore

$$f_{12}+if_{22} = \frac{i}{\pi}\int_\gamma \frac{\bar{F}(1/\sigma)}{\sigma-\zeta}\,d\sigma, \tag{11.5}$$

$$f_{12}-if_{22} = -\frac{i}{\pi}\int_\gamma \frac{F(\sigma)d\sigma}{\sigma-\zeta}. \tag{11.6}$$

If we now put

$$f_1 = \tfrac{1}{12}(1+2\sigma)z^3+f_{11}+(1+\sigma)f_{12}, \tag{11.7}$$

$$f_2 = \tfrac{1}{12}(1+2\sigma)iz^3+f_{21}+(1+\sigma)f_{22}, \tag{11.8}$$

we see that f_1 and f_2 are analytic functions of z within C, whose imaginary parts take up values which are respectively the imaginary and the real parts of the right-hand side of (11.1). Therefore f_1 and f_2 solve the problem of flexure for the boundary C. We will call the four functions f_{11}, f_{12}, f_{21}, f_{22} the canonical complex flexure functions for the boundary.

From (11.5) and (11.6), we get

$$\frac{d}{d\zeta}(f_{12}+if_{22}) = \frac{i}{\pi}\int_\gamma \frac{\bar{F}(1/\sigma)}{(\sigma-\zeta)^2}\,d\sigma,$$

$$\frac{d}{d\zeta}(f_{12}-if_{22}) = -\frac{i}{\pi}\int_\gamma \frac{F(\sigma)}{(\sigma-\zeta)^2}\,d\sigma.$$

Since $\bar{F}(1/\sigma)/(\sigma-\zeta)$ and $F(\sigma)/(\sigma-\zeta)$ are single-valued on γ, and therefore

$$\int_\gamma \frac{d}{d\sigma}\left[\frac{\bar{F}(1/\sigma)}{\sigma-\zeta}\right] d\sigma = 0, \quad \int_\gamma \frac{d}{d\sigma}\left[\frac{F(\sigma)}{\sigma-\zeta}\right] d\sigma = 0,$$

we find that

$$\frac{d}{d\zeta}(f_{12}+if_{22}) = \frac{i}{\pi}\int_\gamma \frac{d\bar{F}(1/\sigma)}{d\sigma}\cdot\frac{d\sigma}{\sigma-\zeta},$$

$$\frac{d}{d\zeta}(f_{12}-if_{22}) = -\frac{i}{\pi}\int_\gamma \frac{dF(\sigma)}{d\sigma}\cdot\frac{d\sigma}{\sigma-\zeta}.$$

From the value of $F(\zeta)$ as given by (11.4), we get

$$\frac{d\bar{F}(1/\sigma)}{d(1/\sigma)} = \frac{1}{2}[\omega(\sigma)]^2\bar{\omega}'(1/\sigma), \quad \frac{dF(\sigma)}{d\sigma} = \frac{1}{2}[\bar{\omega}(1/\sigma)]^2\omega'(\sigma),$$

where $\omega'(\zeta)$ denotes the derivative of $\omega(\zeta)$ with respect to ζ. Hence

$$\frac{d}{d\zeta}(f_{12}+if_{22}) = -\frac{i}{2\pi}\int_\gamma \frac{[\omega(\sigma)]^2\bar{\omega}'(1/\sigma)}{\sigma^2(\sigma-\zeta)}\,d\sigma,$$

$$\frac{d}{d\zeta}(f_{12}-if_{22}) = -\frac{i}{2\pi}\int_\gamma \frac{[\bar{\omega}(1/\sigma)]^2\omega'(\sigma)}{\sigma-\zeta}\,d\sigma,$$

and therefore

$$f_{12}+if_{22} = -\frac{i}{2\pi}\int^\zeta d\zeta\int_\gamma \frac{[\omega(\sigma)]^2\bar{\omega}'(1/\sigma)}{\sigma^2(\sigma-\zeta)}\,d\sigma, \tag{11.9}$$

$$f_{12}-if_{22} = -\frac{i}{2\pi}\int^\zeta d\zeta\int_\gamma \frac{[\bar{\omega}(1/\sigma)]^2\omega'(\sigma)}{(\sigma-\zeta)}\,d\sigma. \tag{11.10}$$

12. Let us now evaluate N_1, N_2. If we put

$$\Psi_1 = \psi_1 - \tfrac{1}{12}(1+2\sigma)(3x^2y-y^3), \quad \Psi_2 = \psi_2 - \tfrac{1}{12}(1+2\sigma)(x^3-3xy^2), \tag{12.1}$$

we see from (11.7) and (11.8) that Ψ_1, Ψ_2 are the imaginary parts of

$$f_{11}+(1+\sigma)f_{12}, \quad f_{21}+(1+\sigma)f_{22}$$

respectively. We find that

$$N_1 = \frac{1}{4}(1-2\sigma)\iint_R y(x^2+y^2)\,dx\,dy - \iint_R\left(x\frac{\partial\Psi_1}{\partial x}+y\frac{\partial\Psi_1}{\partial y}\right)dx\,dy, \tag{12.2}$$

$$N_2 = -\frac{1}{4}(1-2\sigma)\iint_R x(x^2+y^2)\,dx\,dy - \iint_R\left(x\frac{\partial\Psi_2}{\partial x}+y\frac{\partial\Psi_2}{\partial y}\right)dx\,dy. \tag{12.3}$$

As in the case of N_0, we can calculate the second integrals on the right-hand sides of (12.2) and (12.3). They are

$$J_{11} = \frac{1}{2}\mathcal{R}\int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)\,df_{11}(\zeta) + \frac{1}{2}(1+\sigma)\mathcal{R}\int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)\,df_{12}(\zeta), \tag{12.4}$$

and

$$J_{21} = \frac{1}{2}\mathcal{R}\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta)df_{21}(\zeta) + \frac{1}{2}(1+\sigma)\mathcal{R}\int_{\gamma} \omega(\zeta)\bar{\omega}(1/\zeta)df_{22}(\zeta), \tag{12.5}$$

respectively. Again, writing

$$J_{12} = \iint_{R} y(x^2+y^2)\,dxdy = -\frac{1}{4}\int_{C}(x^2+y^2)^2dx,$$

$$J_{22} = \iint_{R} x(x^2+y^2)\,dxdy = \frac{1}{4}\int_{C}(x^2+y^2)^2dy,$$

we get

$$J_{12} - iJ_{22} = -\frac{1}{4}\int_{C} z^2\bar{z}^2dz = -\frac{1}{4}\int_{\gamma}[\omega(\zeta)\bar{\omega}(1/\zeta)]^2d\omega(\zeta). \tag{12.6}$$

In terms of J_{11}, J_{21}, J_{12}, J_{22}, we have

$$N_1 = J_{11} + \tfrac{1}{4}(1-2\sigma)J_{12}, \quad N_2 = J_{21} - \tfrac{1}{4}(1-2\sigma)J_{22}. \tag{12.7}$$

When N_1, N_2 are known, the position of the centre of flexure is determined from (8.8), the moments and products of inertia being given by (9.6) and (9.7) after putting $\alpha = 0$.

13. The stresses at any point of the cross-section are given by

$$X_z - iY_z = \frac{f'(\zeta)}{\omega'(\zeta)} + \frac{1}{2}ik_0\bar{\omega}(\zeta) + \frac{1}{2(1+\sigma)}[k_1\{\sigma y^2-(1+\sigma)x^2\} - ik_2\{\sigma x^2-(1+\sigma)y^2\}], \tag{13.1}$$

where

$$f = -\tfrac{1}{2}k_0f_0 + \tfrac{1}{2}(k_1f_1+k_2f_2)/(1+\sigma) \tag{13.2}$$

and f_0, f_1, f_2 are given by (10.1), (11.2), (11.3), (11.7), (11.8), (11.9) and (11.10).

14. As an application, let us consider a cylinder whose cross-section is bounded by the cardiod,

$$r = 2c(1-\cos\theta).$$

The problem for this boundary has been treated by many authors among whom are Mushtari (1933, 1935), Shepherd (1936), Stevenson (1938), Morris (1939) and Sokolnikoff (1946).

The function which maps the region within the cardiod on the unit circle $|\zeta| \leqq 1$ is

$$z = -c(1-\zeta)^2.$$

We have

$$I_0 = 6\pi c^2, \quad z_0 = -\tfrac{5}{3}c.$$

Transferring the origin to the point z_0, the mapping function becomes

$$z = \omega(\zeta) = \tfrac{1}{3}c(2+6\zeta-3\zeta^2). \tag{14.1}$$

The coordinate axes are the principal axes at the centroid.

Evaluating with the help of Cauchy's residue theorem, we get

$$\int_{\gamma}\omega(\zeta)[\bar{\omega}(1/\zeta)]^2d\omega(\zeta) = \frac{220}{8}i\pi c^4, \tag{14.2}$$

$$\int_\gamma [\omega(\zeta)]^2\bar{\omega}(1/\zeta)d\omega(\zeta) = -\frac{16}{3}i\pi c^4. \tag{14.3}$$

Then (9.6) and (9.7) give

$$I_{11}+I_{22} = \tfrac{55}{3}\pi c^4, \quad I_{11}-I_{22} = -\tfrac{8}{3}\pi c^4,$$

so that

$$I_{11} = \tfrac{47}{6}\pi c^4, \quad I_{22} = \tfrac{21}{2}\pi c^4. \tag{14.4}$$

To evaluate the integral (10.1), *i.e.*, the integral

$$f_0 = \frac{c^2}{18\pi}\int_\gamma\left(-\frac{6}{\sigma^2}-\frac{6}{\sigma}+49-6\sigma-6\sigma^2\right)\frac{d\sigma}{\sigma-\zeta},$$

we observe that we can take the integral over a concentric circle Σ of large radius, because the integrand has no singularity between γ and Σ. Expanding in powers of $1/\sigma$, we see that the integrand has a term in $1/\sigma$ whose coefficient is

$$49-6\zeta-6\zeta^2.$$

Therefore

$$f_0 = \tfrac{1}{9}ic^2(49-6\zeta-6\zeta^2). \tag{14.5}$$

Since

$$\int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)df_0(\zeta) = -\frac{2ic^4}{27}\int_\gamma\left(-\frac{6}{\zeta^2}-\frac{6}{\zeta}+49-6\zeta-6\zeta^2\right)(1+2\zeta)d\zeta = -\frac{8}{3}\pi c^4, \tag{14.6}$$

we get from (10.2), (14.2) and (14.6),

$$N_0 = 17\pi c^4. \tag{14.7}$$

Proceeding with the integrals (11.2), (11.3) in the same manner as we did with (10.1), we get

$$f_{11}+if_{21} = \tfrac{1}{18}c^3(100\zeta-65\zeta^2-6\zeta^3+6\zeta^4),$$

$$f_{11}-if_{21} = \tfrac{2}{9}c^3(4\zeta+\zeta^2),$$

omitting constants. Hence

$$f_{11} = \tfrac{1}{36}c^3(116\zeta-61\zeta^2-6\zeta^3+6\zeta^4), \tag{14.8}$$

$$f_{21} = -\tfrac{1}{12}ic^3(28\zeta-23\zeta^2-2\zeta^3+2\zeta^4). \tag{14.9}$$

Proceeding as before, we find

$$\int_\gamma \frac{[\omega(\sigma)]^2\bar{\omega}'(1/\sigma)}{\sigma^2(\sigma-\zeta)}d\sigma = \frac{4}{3}i\pi c^3(20-15\zeta+3\zeta^2),$$

$$\int_\gamma \frac{[\bar{\omega}(1/\sigma)]^2\omega'(\sigma)}{\sigma-\zeta}d\sigma = -\frac{16}{9}i\pi c^3(5+\zeta).$$

Then (11.9) and (11.10) give

$$f_{12}+if_{22} = \tfrac{1}{3}c^3(40\zeta-15\zeta^2+2\zeta^3),$$

$$f_{12}-if_{22} = -\tfrac{4}{9}c^3(10\zeta+\zeta^2),$$

whence

$$f_{12} = \tfrac{1}{18}c^3(80\zeta-49\zeta^2+6\zeta^3), \tag{14.10}$$

$$f_{22} = -\tfrac{1}{18}ic^3(160\zeta-41\zeta^2+6\zeta^3). \tag{14.11}$$

From symmetry about the x-axis, we see that $N_1 = 0$. Now

$$\int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)df_{21}(\zeta) = 2\pi c^5,$$

$$\int_\gamma \omega(\zeta)\bar{\omega}(1/\zeta)df_{22}(\zeta) = -\frac{52}{9}\pi c^5,$$

so that we get from (12.5),

$$J_{21} = -\tfrac{1}{9}(17+26\sigma)\pi c^5. \tag{14.12}$$

Evaluating the integral (12.6), we get

$$J_{12} - iJ_{22} = -\tfrac{4}{9}i\pi c^5,$$

and therefore

$$J_{22} = \tfrac{4}{9}\pi c^5. \tag{14.13}$$

Substituting in (12.7), we get

$$N_2 = -\tfrac{2}{3}(3+4\sigma)\pi c^5. \tag{14.14}$$

The position of the centre of flexure is given by

$$x_f = -\frac{2(3+4\sigma)c}{63(1+\sigma)}, \quad y_f = 0. \tag{14.15}$$

Since $\omega'(\zeta) = 0$ when $\zeta = 1$, *i.e.*, at the point on γ which corresponds to the cusp on the cardiod, the stress becomes infinite at the cusp unless $f'(\zeta)$ also vanishes for $\zeta = 1$. But of the derivatives of the three functions f_0, f_1, f_2, only that of f_1 vanishes when $\zeta = 1$, so that except in the case of flexure of the cylinder by a load along the axis of symmetry, an infinity of stress appears at the cusp. It is to be pointed out, however, that all other results of the solution remain unaffected.

DEPARTMENT OF APPLIED MATHEMATICS,
CALCUTTA UNIVERSITY.

References

Griffith, A. A. and Taylor, G. I., 1921, *Rep Memor Aero. Res. Comm., Lond.*, No. 389, Nov., 1917 in *Tech. Rep. Adv. Comm. Aero*, *Lond*, 1917-18. (publ. 1921), 3, 950.

Love, A. E. H., 1927, *Mathematical theory of Elasticity* (4th. ed.), Cambridge.

Morris, R. M., 1939, *Proc. Lond. Math. Soc.* (2), 46, 81.

Muschelisvili, N. I., 1929, *Atti della reale accademia nazionale dei Lincei* (6), 9, 295.

Mushtari, N. M., 1938, *Transactions of Kazan Aviational Institute*, 1, 17. (cited by Sokolnikoff, 1946, p. 253).

———, 1935, *Trans. KIIKS*, 1, 53. (cited by Sokolnikoff, 1946, p. 253).

Nemenyi, P., 1921, *Zeit. angew. Math. Mech.*, 1, 89.

Shepherd, W. M., 1936, *Proc. Roy. Soc., Lond* (A), 154, 500.

Sokolnikoff, I. S., 1946, *Mathematical Theory of Elasticity*, New York, p. 244.

Stevenson, A. C., 1938, *Phil. Trans. Roy. Soc*, *Lond.* (A), 237, 161.

Timoshenko, S., 1913, *Bull. Inst. Engineers of Ways of communications*, St. Petersburg. (cited by Timoshenko, 1922).

———, 1922, *Proc. Lond. Math. Soc.* (2), 20, 398.

Villat, H., 1938, *Mecanique des Fluides*, 2ᵉ édition, Paris.

ON A SPECIAL CASE OF THE DISTRIBUTION LAW OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE

By
M. C. Chakrabarti

(*Communicated by Prof. N. M. Basu—Received January 8, 1947*)

1. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a normal population having mean zero and standard deviation equal to σ. Then the statistic

$$\delta^2 = \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 / (n-1)$$

is called the mean square successive difference and on account of the importance attached to it by ballisticians, several papers have appeared in recent years in the *Annals of Mathematical Statistics*, dealing with the distribution law and moments of this and an allied statistic. Neumann, Kent, Bellinson and Hart (1941) in a joint paper have established the distribution law of δ^2 in an explicit form in the special case $n = 3$ by using algebraic transformations and hyperspace geometry. The object of this note is to obtain the same distribution law by the method of characteristic functions. Incidentally, an alternative form of the distribution law involving Laguerre polynomials has also been obtained. Lastly, the moments of the distribution law of δ^2 have been calculated by making use of a known integral involving Bessel function.

2. *The distribution law of δ^2 in the special case $n = 3$*: Here

$$\delta^2 = \{(x_2 - x_1)^2 + (x_3 - x_2)^2\}/2.$$

The joint distribution law of x_1, x_2, x_3 is

$$p(x_1, x_2, x_3) = \frac{1}{\{\sqrt{(2\pi)}.\sigma\}^3} \exp\left[-(x_1^2 + x_2^2 + x_3^2)/2\sigma^2\right].$$

The characteristic function of the distribution-law of δ^2 is

$$\varphi(t) = \frac{1}{\{\sqrt{(2\pi)}.\sigma\}^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(it\delta^2).\ \exp\left[-(x_1^2 + x_2^2 + x_3^2)/2\sigma^2\right] dx_1 dx_2 dx_3$$

$$= \frac{1}{\{\sqrt{(2\pi)}.\sigma\}^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\left[\left(\frac{1}{2\sigma^2} - \frac{it}{2}\right)x_1^2 + \left(\frac{1}{2\sigma^2} - it\right)x_2^2 + \left(\frac{1}{2\sigma^2} - \frac{it}{2}\right)x_3^2 \right.\right.$$
$$\left.\left. + itx_1x_2 + itx_2x_3\right]\right\} dx_1 dx_2 dx_3$$

$$= \frac{1}{\{\sqrt{(2\pi)}.\sigma\}^3} \frac{(\sqrt{\pi})^3}{\begin{vmatrix} \frac{1}{2\sigma^2} - \frac{it}{2}, & \frac{it}{2}, & 0 \\ \frac{it}{2}, & \frac{1}{2\sigma^2} - it. & \frac{it}{2} \\ 0, & \frac{it}{2}, & \frac{1}{2\sigma^2} - \frac{it}{2} \end{vmatrix}^{\frac{1}{2}}} = \frac{1}{\sqrt{[(1 - 3it\sigma^2)(1 - it\sigma^2)]}}.$$

Therefore the distribution law of δ^2 is

$$p(\delta^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-it\delta^2)}{\sqrt{[(1-3it\sigma^2)(1-it\sigma^2)]}}\,dt. \tag{1}$$

Putting $z = 2-3\sigma^2 it$, we get

$$p(\delta^2) = \frac{\exp(-\frac{2}{3}\delta^2/\sigma^2)}{\sigma^2\sqrt{3}.2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\exp(\delta^2.z/3\sigma^2)}{\sqrt{(z^2-1)}}\,dz. \tag{2}$$

Take the contour consisting of

(*A*) the line $R(z) = 2$ from the point $2-i\sqrt{(R^2-4)}$ to the point $2+i\sqrt{(R^2-4)}$;

(*B*) the part of the circle $|z| = R$ to the left of the line $R(z) = 2$ taken positively ;

(*C*) the circle $|z-1| = \epsilon$ taken negatively ;

(*D*) the circle $|z+1| = \epsilon$ taken negatively ;

(*E*) the parts of the real axis from $-1+\epsilon$ to $1-\epsilon$ and $1+\epsilon$ to 2 each taken twice in opposite directions.

Since $\exp(z\delta^2/3\sigma^2)/\sqrt{(z^2-1)}$ is single-valued and analytic inside and on this contour, we have

$$0 = \int_{2-i\sqrt{(R^2-4)}}^{2+i\sqrt{(R^2-4)}} \frac{\exp(z\delta^2/3\sigma^2)}{\sqrt{(z^2-1)}}\,dz + \left\{\left[\int_{(B)} + \int_{(C)} + \int_{(D)} + \int_{2}^{1+\epsilon} + \int_{1+\epsilon}^{2}\right]\frac{\exp(z\delta^2/3\sigma^2)}{\sqrt{(z^2-1)}}\,dz\right\}$$

$$+ \int_{1-\epsilon}^{-1+\epsilon} \frac{\exp(z\delta^2/3\sigma^2)}{\sqrt{(z^2-1)}.\exp(i3\pi/2)}\,dz + \int_{-1+\epsilon}^{1+\epsilon} \frac{\exp(z\delta^2/3\sigma^2).}{\sqrt{(z^2-1)}.\exp(i\pi/2)}\,dz.$$

The fifth and sixth integrals cancel each other; when $R\to\infty$ and $\epsilon\to 0$, it can be easily verified that the second, third and fourth integrals tend to zero. Hence, in the limit,

$$\int_{2-i\infty}^{2+i\infty} \frac{\exp(z\delta^2/3\sigma^2)}{\sqrt{(z^2-1)}}\,dz = 2i\int_{-1}^{+1} \frac{\cos(i\delta^2 z)}{\sqrt{(z^2-1)}}\,dz = 2i\pi J_0(i\delta^2/3\sigma^2).$$

where $J_0(x)$ is the Bessel function of order zero. [See the corollary in p. 366, Whittaker and Watson (1935)].

Therefore from (2), we get

$$p(\delta^2) = \frac{\exp(-\frac{2}{3}\delta^2/\sigma^3)}{\sigma^2\sqrt{3}}\,J_0(i\delta^2/3\sigma^3). \tag{3}$$

3. *An alternative form of the distribution law* : We have from (1)

$$p(\delta^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-it\delta^2)}{\sqrt{[(1-2it\sigma^2)^2+t^2\sigma^4]}}\,dt.$$

Since

$$|(t^2\sigma^4)/(1-2it\sigma^2)^2| = \frac{t^2\sigma^4}{1+4t^2\sigma^4} < \frac{1}{4},$$

the denominator can be written as

$$\sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{t^{2k}\sigma^{4k}}{(1-2it\sigma^2)^{2k+1}}$$

which is uniformly convergent for all real values of t. Since $|\exp(-it\delta^2)| \leq 1$, multiplication by $\exp(-it\delta^2)$ does not affect the uniform convergence of the above series and integrating term by term we get

$$p(\delta^2) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\binom{-1/2}{k}\sigma^{4k}\int_{-\infty}^{\infty}\frac{t^{2k}\exp(-it\delta^2)}{(1-2it\sigma^2)^{2k+1}}\,dt. \tag{4}$$

But it is well-known that

$$\int_{-\infty}^{\infty}\frac{\exp(-itu)}{(1/2\sigma^2-it)^{2k+1}}\,dt = \frac{2\pi}{\Gamma(2k+1)}u^{2k}\exp(-u/2\sigma^2), \text{ for } u > 0.$$

Differentiating $2k$-times with respect to u, we get

$$\int_{-\infty}^{\infty}\frac{t^{2k}\exp(-itu)}{(1/2\sigma^2-it)^{2k+1}}\,dt = \frac{2\pi}{\Gamma(2k+1)}(-1)^k\left(\frac{d}{du}\right)^{2k}[u^{2k}\exp(-u/2\sigma^2)]$$

$$= \frac{2\pi}{\Gamma(2k+1)}(-1)^k\exp(-u/2\sigma^2).L_{2k}(u/2\sigma^2) \tag{5}$$

where $L_{2k}(x)$ is the Laguerre polynomial of degree $2k$.

Therefore from (4) and (5), we get

$$p(\delta^2) = \sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{(-1)^k}{2^{2k+1}\sigma^2}\cdot\frac{1}{(2k)!}\exp(-\delta^2/2\sigma^2).L_{2k}(\delta^2/2\sigma^2)$$

$$= \frac{\exp(-\delta^2/2\sigma^2)}{2\sigma^2}\sum_{k=0}^{\infty}\frac{L_{2k}(\delta^2/2\sigma^2)}{2^{4k}(k!)^2}. \tag{6}$$

4. *Moments of the distribution law*: From the expression for the characteristic function, we can deduce easily

$$\mu_k' = \sigma^{2k}\sum_{s=0}^{k}\binom{k}{s}\left(\frac{1}{2}\right)_s\left(\frac{1}{2}\right)_{k-s}3^s \tag{7}$$

where μ_k' denotes the k-th moment about the origin and the symbol $(a)_s$ denotes $a(a+1)\ldots(a+s-1)$. A still more elegant formula can be obtained from (6). In fact, we have

$$\mu_k' = \frac{1}{2\sigma^2}\sum_{r=0}^{\infty}\frac{1}{2^{4r}(r!)^2}\int_0^{\infty}\delta^{2k}\exp(-\delta^2/2\sigma^2).L_{2r}(\delta^2/2\sigma^2)\,d(\delta^2)$$

$$= (2\sigma^2)^k\,k!\sum_{r=0}^{[k/2]}\frac{1}{2^{4r}(r!)^2}k(k-1)\ldots(k-2r+1) = (2\sigma^2)^k.k!\sum_{r=0}^{[k/2]}\frac{(-k)_{2r}}{2^{4r}(r!)^2}. \tag{8}$$

From (3), we can get another expression for μ_k'. Writing $u = \delta^2$,

$$\mu_k' = \int_0^{\infty}u^k\frac{\exp(-2u/3\sigma^2)}{\sigma^2\sqrt{3}}J_0(iu/3\sigma^2)\,du = \frac{1}{\sigma^2\sqrt{3}}\cdot\frac{k!}{(1/3\sigma^2)^{(k+1)/2}}\cdot{}_2F_1[\tfrac{1}{2}(k+1),\ -\tfrac{1}{2}k,\ 1,\ -\tfrac{1}{3}]$$

$$= \sigma^{2k}.3^{k/2}k!\sum_{n=0}^{\infty}\frac{(\frac{1}{2}\{k+1\})_n(-k/2)_n}{n!\,n!}(-\tfrac{1}{3})^n$$

[*Cf.* Watson (1944), p 385]. When $k = 2r+1$

$$\mu'_{2r+1} = \sigma^{4r+2}3^{r+1/2}(2r+1)!\sum_{n=0}^{\infty}\frac{(r+1)_n(-\tfrac{1}{2}\{2r+1\})_n}{n!\,n!}(-\tfrac{1}{3})^n. \tag{9}$$

Remembering that

$$(1+x)^{k/2} = \sum_{n=0}^{\infty}\binom{k/2}{n}x^n, \quad |x|<1,$$

we get easily, for a positive integer μ,

$$\sum_{n=0}^{\infty}(\mu+n)(\mu+n-1)\ldots(n+1)\binom{k/2}{n}x^n = \left(\frac{d}{dx}\right)^{\mu}[x^{\mu}(1+x)^{k/2}].$$

Hence from (9)

$$\begin{aligned}\mu'_{2r+1} &= \sigma^{4r+2}3^{r+\frac{1}{2}}\frac{(2r+1)!}{r!}\sum_{n=0}^{\infty}(r+n)(r+n-1)\ldots(n+1)\binom{\tfrac{1}{2}(2r+1)}{n}\frac{1}{3^n}\\ &= \sigma^{4r+2}3^{r+\frac{1}{2}}\frac{(2r+1)!}{r!}\left[\left(\frac{d}{dx}\right)^{r}\left\{x^r(1+x)^{(2r+1)/2}\right\}\right]_{x=\frac{1}{3}}\\ &= \sigma^{4r+2}\frac{(2r+1)!}{r!}\sum_{n=0}^{r}\binom{r}{n}(r-n+1)_n(\tfrac{3}{2}+n)_{r-n}2^{2n+1}.\end{aligned} \tag{10}$$

When $k = 2r$, we get from the formula for μ'_k,

$$\mu'_{2r} = \sigma^{4r}(2r)!\sum_{n=0}^{r}\frac{(\tfrac{1}{2}\{2r+1\})_n(-r)_n}{n!\,n!}(-\tfrac{1}{3})^n. \tag{11}$$

It will be found that for practical computation, formula (8) is the best. Formulæ (10) and (11) are to be preferred to formula (7) as the latter contains about double the number of terms. Below are given the first six moments:

$\mu'_1 = 2\sigma^2$ $\mu'_2 = 9\sigma^4$
$\mu'_3 = 66\sigma^6$ $\mu'_4 = 681\sigma^8$
$\mu'_5 = 9090\sigma^{10}$ $\mu'_6 = 148905\sigma^{12}$

DEPARTMENT OF MATHEMATICS & STATISTICS,
DACCA UNIVERSITY.

References

Von Neumann, J., Kent, R. H., Bellinson, H. R., and Hart, B. I., (1941), *Annals of Mathematical Statistics*, 12, 153-162.

Watson, G. N., (1944), *Theory of Bessel functions*, (Second Edition), 385.

Whittaker and Watson, (1935), *Modern Analysis*, (Fourth Edition), 366.

ON (C, 1)-CONVERGENT INTEGRALS AND THEIR APPLICATION TO MATHEMATICAL PHYSICS

By

P. K. Ghosh

(*Received January 29, 1947*)

Introduction

The object of this paper is to develop a theory of infinite integrals which oscillate in the ordinary sense, with a view to use these integrals in the solution of differential equations, so that they may be useful in Mathematical Physics. With the help of these integrals we have treated Rutherford Scattering by Born's Method and obtained the result without any trick unlike Wentzel (1927). We have also evaluated a very important infinite integral used by Bethe (1930). Bethe's evaluation of this integral is far from being justifiable.

Mathematical Theory

We know that if a function $a(x)$ integrable-R tends to l as x tends to infinity, then

$$\int_0^x a(x)dx \Big/ x$$

also tends to the same limit as x tends to infinity. If $a(x)$ diverges definitely, then

$$\int_0^x a(x)dx \Big/ x$$

also diverges definitely. We now define $(C, 1)$-limit of $a(x)$ as follows:

Definition. *If*

$$\int_0^x a(x)dx \Big/ x$$

tends to a limit l as $x \to \infty$, we call this limit the $(C, 1)$-limit of $a(x)$ as $x \to \infty$.

Example: In the ordinary sense $\sin x$ oscillates as $x \to \infty$, but the $(C, 1)$-limit of $\sin x$ as $x \to \infty$, *viz.*,

$$(C, 1)\text{-}\lim_{x \to \infty} \sin x = \lim_{x \to \infty} \int_0^x \sin x \, dx \Big/ x = 0.$$

In case the function $a(x)$ is not defined, say, from 0 to b, we consider another function $A(x)$ such that

$$A(x) = 0, \qquad 0 \leq x \leq b,$$

$$A(x) = a(x), \quad x > b,$$

and take the limit of

$$\int_0^x A(x)dx \Big/ x$$

(provided the limit exists) as x tends to ∞ as the $(C, 1)$-limit of $a(x)$ as $x \to \infty$.

Now let us consider the integral

$$F(x) = \int_c^x f(x)dx. \tag{1}$$

If $(C, 1)\text{-}\lim\limits_{x \to \infty} F(x)$ exists, then we say that the integral on the right of (1) is $(C, 1)$-convergent and we take this limit as the value of the infinite integral. We denote this integral by

$$\overset{\bullet}{\int_c^\infty} f(x)dx$$

with a dot over the sign of integration.

Example: Let us consider the integral

$$F(x) = \int_0^x \sin x \, dx = 1 - \cos x.$$

In the ordinary sense $F(x)$ oscillates as $x \to \infty$, but

$$\overset{\bullet}{\int_0^\infty} \sin x \, dx = 1.$$

This concept of convergence of an infinite integral (rather a more generalised concept) has come into existence in connection with investigations regarding Fourier Integrals [*Cf.* Titchmarsh (1937)]. There the term 'summability' is used instead of 'convergence'. The discussion which follows will, I hope, sufficiently justify this slight change in terminology.

UNIFORM CONVERGENCE

These infinite integrals may also contain parameters. *Let*

$$F(x, \alpha) = \int_0^x dx \int^x f(y, \alpha)\, dy \Big/ x. \tag{2}$$

If now $F(x, \alpha)$ *converges uniformly in an interval* $A \leq \alpha \leq B$ *as* $x \to \infty$, *we say that the integral*

$$\overset{\bullet}{\int_c^\infty} f(x, \alpha)\, dx \tag{3}$$

converges uniformly in the interval $A \leq \alpha \leq B$.

We now proceed to study some important properties of $(C, 1)$-convergent integrals containing a parameter.

CONTINUITY WITH RESPECT TO A PARAMETER

Theorem 1: *If $f(x, \alpha)$ is continuous in the region $c \leq x \leq \infty$, $A \leq \alpha \leq B$ and if the integral*

$$\int^{\infty} f(x, \alpha)\, dx$$

converges uniformly in the interval $A \leq \alpha \leq B$, then the integral defines a function

$$\varphi(\alpha) = \int_c^{\infty} f(x, \alpha)\, dx$$

continuous in that interval.

Proof. We take any point α_0 in the interval $A \leq \alpha \leq B$. As the integral is uniformly convergent,

$$\varphi(\alpha_0) = \int_0^{x'} dx \int^{x} f(y, \alpha_0) dy \Big/ x' + R_{x'}(\alpha_0)$$

and

$$\varphi(\alpha_0 + h) = \int_0^{x'} dx \int_c^{x} f(y, \alpha_0 + h) dy \Big/ x' + R_{x'}(\alpha_0 + h)$$

where both $|R_{x'}(\alpha_0)| < \epsilon$ and $|R_{x'}(\alpha_0 + h) < \epsilon$ when $x' > X(\epsilon)$.

$$\varphi(\alpha_0 + h) - \varphi(\alpha_0) = \int_0^{x'} dx \int_c^{x} [f(y, \alpha_0 + h) - f(y, \alpha_0)] dy \Big/ x' + R_{x'}(\alpha_0 + h) - R_{x'}(\alpha_0).$$

Hence

$$|\varphi(\alpha_0 + h) - \varphi(\alpha_0)| \leq \left| \int_0^{x'} dx \int_c^{x} [f(y, \alpha_0 + h) - f(y, \alpha_0)] dy \Big/ x' \right| + |R_{x'}(\alpha_0 + h)| + |R_{x'}(\alpha_0)|.$$

Now we keep x' fixed. Then, since

$$\int_0^{x'} dx \int_c^{x} f(y, \alpha) dy \Big/ x'$$

is a continous function of α in the interval $A \leq \alpha \leq B$, it follows that

$$\left| \int_0^{x'} dx \int_0^{x} [f(y, \alpha_0 + h) - f(y, \alpha_0)] dy \Big/ x' \right| < \varepsilon, \quad |h| < \delta$$

Hence

$$|\varphi(\alpha_0 + h) - \varphi(\alpha_0)| < 3\varepsilon, \quad |h| < \delta$$

P 17589

which proves the proposition.

INTEGRATION WITH RESPECT TO A PARAMETER: REVERSAL OF THE ORDER OF INTEGRATION

Theorem 2. *If the function* $f(x, \alpha)$ *is continuous in the region* $c \leq x < \infty$, $A \leq \alpha \leq B$, *and if the integral*

$$\int_c^\infty f(x, \alpha)\, dx$$

converges uniformly in the interval $A \leq \alpha \leq B$:

$$\varphi(\alpha) = \int_c^\infty f(x, \alpha) dx$$

then

$$\int_{\alpha_0}^{\alpha_1} \varphi(\alpha) d\alpha = \int_c^\infty dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha) d\alpha$$

or

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_c^\infty f(x, \alpha) dx = \int_c^\infty dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha) d\alpha.$$

Proof. The function $\varphi(\alpha)$ is continuous by Theorem 1 and therefore integrable. Because of uniform convergence

$$\varphi(\alpha) = \int_0 \; dx \int_c^x f(y, \alpha) dy \Big/ x' + R_{x'}(\alpha)$$

where $| R_{x'}(\alpha) | < \epsilon$ when $x' > X(\epsilon)$.

$$\int_{\alpha_0}^{\alpha_1} \varphi(\alpha) d\alpha = \int_{\alpha_0}^{\alpha_1} d\alpha \int_0^{x'} dx \int_c^x f(y, \alpha) dy \Big/ x' + \int_{\alpha_0} \; R_{x'}(\alpha) d\alpha.$$

We keep x' fixed. The order of integration in the first integral on the right can be reversed, so that

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_c^\infty f(x, \alpha) dx - \int_0^{x'} dx \int_c^x dy \int_{\alpha_0}^{\alpha_1} f(y, \alpha) d\alpha \Big/ x' = \int_{\alpha_0}^{\alpha_1} R_{x'}(\alpha) d\alpha.$$

Hence

$$\left| \int_{\alpha_0}^{\alpha_1} d\alpha \int_c^\infty f(x, \alpha) dx - \int_0^{x'} dx \int_c^x dy \int_{\alpha_0}^{\alpha_1} f(y, \alpha) d\alpha \Big/ x' \right| < \epsilon(\alpha_1 - \alpha_0)$$

for $x' > X$. Therefore

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_c^\infty f(x, \alpha) dx = \int_c^\infty dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha) d\alpha.$$

DIFFERENTIATION WITH RESPECT TO A PARAMETER

Theorem 3. *Let the integral*

$$\int_c^\infty f(x, \alpha)\,dx$$

converge in the interval $A \leq \alpha \leq B$:

$$\varphi(\alpha) = \int_c^\infty f(x, \alpha)\,dx.$$

Let $\partial f/\partial\alpha$ *exist and let the function*

$$\partial f/\partial\alpha = f_\alpha(x, \alpha)$$

be continuous in the region $c \leq x < \infty$, $A \leq \alpha \leq B$. *Let the integral*

$$\int_c^\infty f_\alpha(x, \alpha)\,dx$$

converge uniformly in the interval $A \leq \alpha \leq B$.

Then the function $\varphi(\alpha)$ *has a derivative, the derivative is continuous, and it is given by the last integral:*

$$\frac{d\varphi}{d\alpha} = \int_c^\infty f_\alpha(x, \alpha)\,dx.$$

Proof. Let

$$\psi(\alpha) = \int_c^\infty f_\alpha(x, \alpha)\,dx.$$

Then $\psi(\alpha)$ is continuous and therefore integrable. By Theorem 2,

$$\int_A^\alpha \psi(\alpha)\,d\alpha = \int_c^\infty dx \int_A^\alpha \frac{\partial f}{\partial\alpha}\,d\alpha = \int_c^\infty \{f(x, \alpha) - f(x, A)\}\,dx = \int_c^\infty f(x, \alpha)\,dx - \int_c^\infty f(x, A)\,dx.$$

Hence

$$\int_A^\alpha \psi(\alpha)\,d\alpha = \varphi(\alpha) - \varphi(A).$$

From the continuity of $\psi(\alpha)$, it at once follows that

$$\psi(\alpha) = d\varphi/d\alpha$$

The above three theorems proved for only one parameter can be at once generalised for n parameters.

We now proceed to generalise the concept of a volume integral:

$$\int f(x_1, x_2, x_3)dx_1dx_2dx_3 \tag{4}$$

taken over all space. We first integrate over a sphere of radius r, with the origin as centre, so that the integral is a function of r only, *viz.*,

$$\int_{\text{sph}} f(x_1, x_2, x_3)dx_1dx_2dx_3 = F(r). \tag{5}$$

If now the $(C, 1)$-limit of $F(r)$ as $r \to \infty$ exists, we then say that the integral (4) exists and we take the $(C, 1)$-limit of $F(r)$ as r tends to infinity as its value:

$$\dot{\int} f(x_1, x_2, x_3)dx_1dx_2dx_3 = (C, 1)\text{-}\lim_{r \to \infty} F(r). \tag{6}$$

As before we put a dot over the sign of integration to indicate the $(C, 1)$-limit of the integral. Here we have made the origin a special point. We could take any other point as the centre of the sphere of integration and that would lead, in general, to different result. But in the problems which we have in view in the present paper we really have such a special point with some amount of symmetry about it. However, in any case, we shall explicitly mention the position of the centre of the sphere of integration and that will set all doubt at rest.

Now we generalise the concept, when the integrals contain parameters:

$$\int f(\alpha_1, \alpha_2, \ldots \alpha_r;\ x_1, x_2, x_3)dx_1dx_2dx_3 \tag{7}$$

taken over all space. Here we shall not make any special choice of the centre of the sphere of integration. We take any point as the centre of the sphere of integration, and integrate over the sphere and take the $(C, 1)$-limit of this integral as $r \to \infty$. This introduces some arbitrariness in the value of the integral and the integral besides being a function of α's is also a function of the coordinates of the centre of sphere of integration, except when the centre is fixed at the origin or given by the α's. But this arbitrariness can be removed by prescribing the boundary value or form or both of the integral. We could also take instead of spherical regions of integration regions of any other type characterised by only one parameter. That would affect only the form of the integral. This is not surprising because these integrals will be solutions of partial differential equations with respect to the parameters and so long as the conditions of Theorems 1, 2 & 3 are fulfilled, they enjoy to be solutions of different forms We denote this generalised integral with a dot over the sign of integration as before. The generalisation for any number of dimensions can be done immediately.

We now consider some examples. Let us try to evaluate the integral

$$I = \int \frac{\exp\,[i(\mathbf{qr})]}{r}\, d\Omega \tag{8}$$

taken over all space. We take the direction of $\mathbf{q}$ as the polar axis. Now, taken all over a sphere of radius r with the origin as centre

$$I = \int_0^r r^2 dr \int_0^\pi 2 \sin \theta \, d\theta . \, \frac{1}{r} \exp (iqr \cos \theta) = \frac{4\pi}{q} \int_0^r \sin qr \, dr = \frac{4\pi}{q^2} (1 - \cos qr) = F(r). \tag{8'}$$

The $(C, 1)$-limit of $F(r)$ as $r \to \infty$ is $4\pi/q^2$ and we take this as the value of the integral:

$$I = \int \frac{1}{r} \exp [i(\mathbf{qr})] d\Omega = \frac{4\pi}{q^2}. \tag{9}$$

We consider also the differential equation:

$$\Delta \varphi_q(\mathbf{r}) = - 4\pi \exp [i(\mathbf{qr})]. \tag{10}$$

The form suggests that the integral is

$$\varphi_q(\mathbf{r}) = \int \frac{\exp [i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|} d\Omega_R \tag{11}$$

taken over all space. But this integral does not exist in the ordinary sense.

$$\varphi_q(\mathbf{r}) = \int \frac{\exp [i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|} d\Omega_R = \exp [i(\mathbf{qr})] . \int \frac{\exp [i(\mathbf{q}, \mathbf{R}-\mathbf{r})]}{|\mathbf{R}-\mathbf{r}|} d\Omega_R.$$

If we now take the centre of the sphere of integration at the point $\mathbf{r}$, and make the radius of this sphere tend to ∞, then from (9) and (12), we get

$$\varphi_q(\mathbf{r}) = \int \frac{\exp [i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|} d\Omega_R = \frac{4\pi}{q^2} \exp [i(\mathbf{qr})]. \tag{13}$$

That the value $4\pi \exp [i(\mathbf{qr})]/q^2$ satisfies the equation (10) can be verified immediately. But this agreement is not superficial only. The partial derivatives

$$\frac{\partial \varphi_q}{\partial x}, \quad \frac{\partial^2 \varphi_q}{\partial x^2}, \ldots \text{ etc.}$$

are given by

$$\int \frac{\partial}{\partial x} \left(\frac{\exp [i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|} \right) d\Omega_R, \quad \int \frac{\partial^2}{\partial x^2} \left(\frac{\exp [i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|} \right) d\Omega \ \ldots \text{ etc.}$$

respectively.

One might say that

$$\varphi_q' (\mathbf{r}) = \int \frac{\exp [i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|} d\Omega_R$$

where the sphere of integration has the origin as the centre is equally well a solution of (10). We at once see as φ_q and φ_q' are both bounded, they can at most differ by a constant since $\varphi_q - \varphi_q'$ is harmonic over all space. If now the $(C, 1)$-limit of φ_q is prescribed, it is uniquely determined almost everywhere. If we demand that φ_q should behave, as much as possible, as a potential function, say,

$$(C,\,1)\text{-}\lim_{r\to\infty} \varphi_q = 0$$

almost everywhere, then

$$\varphi_q = \frac{4\pi}{q^2} \exp\,[i(\mathbf{qR})]$$

is the only solution.

Now we turn to the integral

$$\int \frac{\exp\,[i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|}\, d\Omega_R$$

taken over all space. The integral does not exist in the ordinary sense. If we take the (C, 1)-limit that too depends on the centre of the spere of integration, so that to fix the value of the integral we require some additional condition. If we take the additional condition as that it should behave as a potential function as much as possible *viz.*, the (C, 1)-limit of the integral is zero almost everywhere, then we find that

$$\int \frac{\exp\,[i(\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|} = \frac{4\pi}{q^2} \exp\,[i(\mathbf{qr})] \tag{14}$$

uniquely.

APPLICATIONS

For the purpose of application we generalise the usual quantum mechanical rule for calculating matrix elements of an operator A, *viz.*,

$$A_{mn} = \int \psi_m^* A \psi_n d\tau \tag{15}$$

instead of

$$A_{mn} = \int \psi_m^* A \psi_n d\tau$$

taken over the entire configuration space where ψ_m and ψ_n are eigenfunctions belonging to the m-th and n-th state respectively. Obviously this generalisation is consistent.

(i) *Scattering by a centre of Force. Born's Approximation.*

Our problem is to calculate the scattering of a beam of particles $[\exp\,(ikz)]$ by a field $V(r)$. We shall obtain an approximate formula.

We have to solve the wave equation

$$\Delta\psi + [k^2 - U(r)]\psi = 0, \tag{16}$$

where $k^2 = 8\pi^2 mE/h^2$, $U(r) = 8\pi^2 mV(r)/h^2$ and where ψ must have the asymptotic form

$$\psi \sim e^{ikz} + r^{-1} e^{ikr} f(\theta), \tag{17}$$

We know that the most general bounded solution of the equation

$$\Delta\psi + k^2\psi = \Phi(\mathbf{r})$$

is

$$\psi = G(\mathbf{r}) - \frac{1}{4\pi}\int \frac{\exp\left[ik\,|\mathbf{r}-\mathbf{r}'|\right]}{|\mathbf{r}-\mathbf{r}'|}\,\Phi(\mathbf{r}')d\tau' \tag{18}$$

where G is the general solution of

$$\Delta G + k^2 G = 0.$$

We can at once generalise (17) in the sense of $(C, 1)$ convergence

$$\psi = G(\mathbf{r}) - \frac{1}{4\pi}\overset{\bullet}{\int} \frac{\exp\left[ik\,|\mathbf{r}-\mathbf{r}'|\right]}{|\mathbf{r}-\mathbf{r}'|}\,\Phi(\mathbf{r}')d\tau'. \tag{18'}$$

Here we take the centre of the sphere of integration at the origin, and we shall see later that this leads to the prescribed asymptotic form.

The general solution ψ of (16) satisfies the integral equation.

$$\psi = G - \frac{1}{4\pi}\overset{\bullet}{\int} \frac{\exp\left[ik\,|\mathbf{r}-\mathbf{r}'|\right]}{|\mathbf{r}-\mathbf{r}'|}\,U(r')\psi(\mathbf{r}')d\tau'. \tag{19}$$

The expression on the right of (18) has to represent an outgoing wave. In order that ψ may have the prescribed asymptotic from (17), we must choose

$$G = e^{ikz}$$

To obtain $f(\theta)$, we require the asymptotic from (18′) for large r. If $\mathbf{n}$ be a unit vector in the direction of $\mathbf{r}$, we have

$$|\mathbf{r}-\mathbf{r}'| \sim r - (\mathbf{n}\mathbf{r}) + \text{terms of the order } 1/r.$$

Hence from (18′)

$$\psi \sim e^{ikz} - r^{-1}e^{ikr}\frac{1}{4\pi}\overset{\bullet}{\int} \exp\left[-ik(\mathbf{n}\mathbf{r}')\right]U(r')\psi(\mathbf{r}')d\tau'. \tag{20}$$

To obtain a formula for $f(\theta)$, we assume that the wave is not much diffracted by the scattering centre, so that we can replace $\psi(\mathbf{r}')$ in the integral (20) by the unperturbed wave function e^{ikz}.

We then obtain from (17) and (20)

$$f(\theta) = -\frac{1}{4\pi}\overset{\bullet}{\int} \exp\left[ik(\mathbf{n}_0-\mathbf{n},\,\mathbf{r}')\right]U(r')d\tau' \tag{21}$$

where $\mathbf{n}_0$ is the unit vector along the z-axis, so that $z = (\mathbf{n}_0\mathbf{r})$.

For the Coulomb field

$$U(r) = \frac{8\pi^2 me}{h^2 r}$$

and hence

$$f(\theta) = -\frac{2\pi me}{h^2}\int \frac{\exp\,[ik(\mathbf{n}_0-\mathbf{n},\,\mathbf{r}')]}{r'}\,d\tau' = -\frac{8\pi^2 me}{h^2}\,\frac{1}{k^2\,|\,\mathbf{n}_0-\mathbf{n}\,|^2} \tag{22}$$

by (14).

Now $|\,\mathbf{n}_0-\mathbf{n}\,| = 2\sin\frac{1}{2}\theta$, therefore

$$f(\theta) = -\frac{4\pi^2 me}{h^2k^2\sin^2\frac{1}{2}\theta}$$

The intensity scattered into the solid angle $d\omega$ is $|\,f(\theta)\,|^2 d\omega$, and this gives the Rutherford Scattering Formula.

Here with the help of $(C, 1)$-convergence of infinite integrals we come to the result without any hitch. But if this idea is not brought in, then the right hand side of (22) oscillates as shown above in (8′). To overcome this difficulty Wentzel (1927) had to use a modified field

$$V = \frac{e}{r}\exp\,(-r/R)$$

instead of the coulomb field and then neglect the modification under certain assumptions. From the mathematical stand-point this is equivalent to the introduction of a "convergence-factor"

$$e^{-pr}, \quad p > 0$$

in the expression

$$\int_0^\infty \sin qr\,dr$$

and after integration let $p \rightarrow 0$. Thus

$$\int_0^\infty dr\; e^{-pr}\sin qr = q/(p^2+q^2) \rightarrow 1/q$$

as $p \rightarrow 0$, and to take this as the value of the integral (Sommerfeld, 1939). This is not justifiable.

(ii) Another example of the application of our theory is furnished by Bethe's investigation regarding the passage of fast particles through matter (Bethe, 1930). There, proceeding by the method of variation of parameters, Bethe had to calculate matrix-elements which involved integrals of the type

$$\int \frac{\exp\,[i(\mathbf{qr})]}{|\,\mathbf{R}-\mathbf{r}\,|}\,d\Omega_R$$

taken over all space. As investigated above this integral does not exist in the ordinary sense. Bethe evaluates it by an unjustifiable interchange of limit processes. In the light of our discussion on (C, 1)-convergent integrals, together with the boundary conditions and generalised rule for calculating the matrix-elements we are at once led to the calculation of integrals of the type

$$\int \frac{\exp\,[i(\mathbf{qr})]}{|\mathbf{R}-\mathbf{r}|}\, d\Omega_R = \frac{4\pi}{q^2} \exp\,[i(\mathbf{qr})]$$

by (14) subject to the conditions laid down there.

No doubt, in our discussion. there has been some arbitrariness associated with the position of the centre of the sphere of integration, but in every one of the applications physical requirements in the form of boundary conditions have decided matters unequivocally. This also brings out the role of boundary conditions in the calculation of semi-convergent multiple integrals.

I have a great pleasure to thank Prof. N. R. Sen, in whose seminar the problem first arose, for helpful criticism when the paper was presented in his seminar. I also thank Prof. S. N. Bose for stimulating discussions regarding the subject matter of the paper.

DEPARTMENT OF APPLIED MATHEMATICS,
UNIVERSITY OF CALCUTTA.

References

Bethe, H., (1930), *Ann. d. Phys.*, 5, 325.

Sommerfeld, A., (1939), *Atombau u. Spektrallinien. Bd II*, 396.

Titchmarsh, E. C., (1937), *Introduction to the Theory of Fourier Integrals*, 26.

Wentzel, G., (1927), *Zeit. f. Phys.*, 40, 590.

CALCUTTA UNIVERSITY CENTRAL LIBRARY

AN IMPORTANT PARTICULAR CASE OF THE PROBLEM OF EQUIVALENCE

By

K. R. Karmarkar

(*Communicated by Prof. V. V. Narlikar—Received January 29, 1947*)

Abstract. A practical method of testing the equivalence of two general spherically symmetrical quadratic forms has been developed.

1. Given two quadratic differential forms which are Riemannian metrics, the question naturally arises whether they are obtainable from each other by a Gaussian transformation of coordinates. This is usually called the problem of equivalence. The general method of solving the problem depends on the properties of the curvature tensor (Oswald Veblen, 1927*a*). An alternative method which has a restricted domain of application depends on the scalar invariants of a quadratic form which are rational functions of the fundamental tensor $g_{\mu\nu}$, the curvature tensor and its covariant derivatives upto any arbitrary order (Oswald Veblen, 1927*b*). These methods have as yet been little applied in particular cases. As far as we know the particular case of the equivalence of two general spherically symmetrical quadratic forms, which is of practical importance in general relativity has not been dealt with elsewhere. We give a simple method suited to the present investigation. The method depends on the scalar invariants of a spherically symmetrical quadratic form.

2. Let us consider the spherically symmetrical form

$$ds^2 = -e^{\lambda}d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu}d\tau^2 \tag{1}$$

where λ and ν are functions of ρ and τ.

Only a coordinate transformation of the type

$$\rho = \bar{\rho}, \quad \tau' = \psi(\bar{\tau}) \tag{2}$$

leaves this structure unaltered. The functions e^{λ} and $\frac{1}{2}\partial\nu/\partial\rho$ remain unaltered under the group of transformations (2). A general spherically symmetrical line-element,

$$ds^2 = -Adr^2 - B(d\theta^2 + \sin^2\theta\, d\phi^2) + 2Cdrdt + Ddt^2, \tag{3}$$

is always reducible to the form (1) except when B is a function of t alone. Suppose (3) is transformed to the form (1). The laws of transformation of tensors give

$$e^{\lambda} = A\left(\frac{\partial r}{\partial\rho}\right)^2 - 2C\frac{\partial r}{\partial\rho}\cdot\frac{\partial t}{\partial\rho} - D\left(\frac{\partial t}{\partial\rho}\right)^2, \tag{3,a}$$

$$\rho^2 = B, \tag{4}$$

$$e^{\nu} = D\left(\frac{\partial t}{\partial\tau}\right)^2 + 2C\frac{\partial r}{\partial\tau}\cdot\frac{\partial t}{\partial\tau} - A\left(\frac{\partial r}{\partial\tau}\right)^2, \tag{5}$$

$$0 = C\left(\frac{\partial r}{\partial\rho}\cdot\frac{\partial t}{\partial\tau} + \frac{\partial r}{\partial\tau}\cdot\frac{\partial t}{\partial\rho}\right) + D\frac{\partial t}{\partial\rho}\cdot\frac{\partial t}{\partial\tau} - A\frac{\partial r}{\partial\rho}\cdot\frac{\partial r}{\partial\tau}. \tag{6}$$

In what follows dashes and dots represent differentiation with respect to r and t respectively. From (4) we obtain

$$1 = \frac{B'}{2B^{1/2}}\frac{\partial r}{\partial \rho} + \frac{\dot{B}}{2B^{1/2}}\frac{\partial t}{\partial \rho}, \tag{7}$$

$$0 = \frac{B'}{2B^{1/2}}\frac{\partial t}{\partial \tau} + \frac{\dot{B}}{2B^{1/2}}\frac{\partial t}{\partial \tau}. \tag{8}$$

Let $m = DB' - C\dot{B}$; $n = CB' + AB$. Eliminating $\partial r/\partial \rho$, $\partial t/\partial \rho$, $\partial r/\partial \tau$, $\partial t/\partial \tau$ between (8), (7), (6), (5), (3a) we get

$$e^{\lambda} = E \equiv 4B(AD + C^2)(B'm - \dot{B}n)^{-1}. \tag{9}$$

Differentiating and subsequently eliminating $\partial^2 r/\partial \rho \partial \tau$ we get

$$\frac{1}{2}\frac{\partial \nu}{\partial \rho} = F \equiv B^{\frac{1}{2}}(B'm - Bn)^{-2}[(-m^2B'' + 2mn\dot{B}' - n^2\dot{B})$$
$$+ B'(mm' + n\dot{m} - 2\dot{n}m) + \dot{B}(n\dot{n} + n'm - 2nm')]. \tag{10}$$

Since e^{λ} and $\frac{1}{2}\partial \nu/\partial \rho$ are invariants for the unique structure (1), it is clear that the expressions E and F given by (9) and (10) are the invariants of the form (3) for the group of transformations

$$r = r(\bar{r}, \bar{t}); \quad t = t(\bar{r}, \bar{t}). \tag{11}$$

The invariants E and F were obtained on the specific assumption that (3) is transformable to (1). If however B is a function of t alone, the same invariants (9) and (10) could be obtained by considering (3) to be transformable to

$$ds^2 = -e^{\lambda}d\rho^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu}d\tau^2, \tag{12}$$

by an exactly similar procedure.

In addition to E and F, B is obviously an invariant for the group of transformations (11).

3. The invariants obtained in the last section will be used to test the equivalence of two given spherically symmetrical quadratic forms. Let the two forms be

$$ds^2 = -Adr^2 - B(d\theta^2 + \sin^2\theta\, d\phi^2) + 2Cdrdt + Ddt^2, \tag{13}$$

and

$$ds^2 = -\bar{A}d\bar{r}^2 - \bar{B}(d\theta^2 + \sin^2\theta\, d\phi^2) + 2\bar{C}d\bar{r}d\bar{t} + \bar{D}d\bar{t}^2. \tag{14}$$

Whenever the forms are equivalent the equations of transformation are consistent with the conditions, $E = \bar{E}$, $F = \bar{F}$ and $B = \bar{B}$ and yet the latter may not always define the transformation. Hence a discussion of particular cases is given below.

Let $\bar{E}$ and $\bar{F}$ be the corresponding expressions for (14). The following cases arise;

(1) B and $\bar{B}$ are functions of r and t and $\bar{r}$ and $\bar{t}$ respectively. The forms are both transformable into (1).

(2) B and $\bar{B}$ are respectively functions of t and $\bar{t}$ only. The forms are not transformable into (1).

(3) B is a function of r and t; $\bar{B}$ is a function of t only.

(4) B is not a constant but a function of t only; $\bar{B}$ is a constant.

(5) $B = \bar{B} = \text{const}$; this is a degenerate case which will not be considered here.

In cases (3) and (4) the forms are not equivalent for obvious reasons. Cases (1) and (2) are dealt with below. For the case (1) the following sub-cases arise, f, φ, ψ, χ being arbitrary functional forms:

$$(a)\quad E = f(B); \quad F = \varphi(B); \quad \bar{E} = \psi(\bar{B}); \quad \bar{F} = \varphi(\bar{B}).$$

$$(b)\quad E = f(B); \quad F = \varphi(B); \quad \bar{E} = f(\bar{B}); \quad \bar{F} = \psi(\bar{B}).$$

$$(c)\quad E = f(B); \quad F = \varphi(B); \quad \bar{E} = \psi(\bar{B}); \quad \bar{F} = \chi(\bar{B}).$$

$$(d)\quad E = f(B); \quad F \neq \varphi(B); \quad \bar{E} = \varphi(\bar{B}).$$

$$(e)\quad E = f(B); \quad \bar{E} \neq \chi(B).$$

The forms are not equivalent in the above cases.

$$(f)\quad E = f(B); \quad F = \varphi(B); \quad \bar{E} = f(\bar{B}); \quad \bar{F} = \varphi(\bar{B}).$$

The forms are equivalent.

$$(g)\quad E = f(B); \quad F \neq \varphi(B); \quad \bar{E} = f(\bar{B}); \quad \bar{F} \neq \chi(\bar{B}).$$

The functional equations $F = \bar{F}$, $B = \bar{B}$ give the transformation if the forms are equivalent. The equivalence may be tested by actual transformation by the above functional equations.

$$(h)\quad E \neq f(B); \quad F = \varphi(B); \quad \bar{E} \neq \psi(\bar{B}); \quad \bar{F} = \varphi(\bar{B}).$$

The equivalence may be tested as in case (g). An exactly similar discussion will apply to the case (2).

The author should like to acknowledge his indebtedness to Professor V. V. Narlikar for the guidance that he has received from him in the preparation of this note.

Department of Mathematics,
Benares Hindu University

References

Veblen, O., (1927a), *Invariants of quadratic Differential Forms*, 67-78.
———, (1927b), *Invariants of quadratic Differential Forms*, 97.

ON RAMANUJAN'S FUNCTION $\tau(n)$ AND THE DIVISOR FUNCTION $\sigma_k(n)$, PART II

By

D. B. Lahiri

(*Communicated by R. C. Bose—Received December 3, 1946*)

1. We have in Part I of this paper (Lahiri, 1946) developed our methods of studying the congruence properties of $\sigma_k(n)$. But what is in a way more interesting is the ease with which the methods can be extended to cover the widely different function, Ramanujan's $\tau(n)$, which is defined by

$$x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} = \sum_1^\infty \tau(n)x^n. \tag{17}$$

Ramanujan's function $\tau(n)$ though widely different from $\sigma_k(n)$ is not entirely unrelated to it as shown by him in his study of the function $\Sigma_{r,s}(n)$. He proved indeed that if r and s are odd integers such that $r+s = 10$ (*i.e.*, for the three pairs of values $r = 1,\ s = 9$; $r = 3,\ s = 7$; $r = 5,\ s = 5$) we have

$$\begin{aligned}\Sigma_{r,s}(n) = {} & \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}.\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}.\sigma_{r+s+1}(n) + \frac{\zeta(1-r)+\zeta(1-s)}{r+s}\,n\sigma_{r+s-1}(n) \\ & + \left\{\frac{\zeta(-r)+\zeta(-s)}{2} - \frac{\zeta(1-r)+\zeta(1-s)}{r+s} - \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}.\frac{\zeta(r+1)\zeta(s+1)}{\zeta(r+s+2)}\right\}\tau(n).\end{aligned} \tag{18}$$

In what follows we shall find it convenient to denote $\sum_1^\infty \tau(n)x^n$ by $\{0\}$.

We have previously considered all products of (u, v)'s which can be expressed as a linear function also of (u, v)'s. We shall now consider those products which require $\{0\}$ in addition, in their linear expressions. For this purpose we shall make use of an extension of Table A(1).

Table A(2)

1. $$P^6 = 1 - \frac{2^{10}.3^5}{5}(5, 6) + \frac{2^9.3^3.5}{7}(4, 7) - 2^7.3^3.5(3, 8) + 2^6.3^2.5^2(2, 9) - \frac{2^5.3^3.11}{5}(1, 10) + \frac{2^4.3^2.5.7.13}{691}(0, 11) - \frac{2^9.3^2}{5.7.691}\{0\},$$

2. $$Q^3 = 1 + \frac{2^4.3^2.5.7.13}{691}(0, 11) + \frac{2^7.3^3.5^3}{691}\{0\},$$

3. $$R^2 = 1 + \frac{2^4.3^2.5.7.13}{691}(0, 11) - \frac{2^6.3^5.7^2}{691}\{0\},$$

4. $P^4Q \;= 1+\dfrac{2^9.3^4}{7}(4,\,7)-2^7.3^4(3,\,8)+2^7.3^2.5(2,\,9)$
$$-\frac{2^6.3^2.11}{5}(1,\,10)+\frac{2^4.3^2.5.7.13}{691}(0,\,11)-\frac{2^9.3^2}{5.7.691}\{0\},$$

5. $P^2Q^2 \;= 1+2^6.3.5(2,\,9)-\dfrac{2^5.3^2.11}{5}(1,\,10)+\dfrac{2^4.3^2.5.7.13}{691}(0,\,11)+\dfrac{2^7.3.97}{5.691}\{0\},$

6. $P^3R \;= 1-2^5.3^4(3,\,8)+2^6.3^2.5(2,\,9)-\dfrac{2^4.3^3.11}{5}(1,\,10)$
$$+\frac{2^4.3^2.5.7.13}{691}(0,\,11)-\frac{2^5.3^2.101}{5.691}\{0\},$$

7. $PQR \;= 1-\dfrac{2^4.3^2.11}{5}(1,\,10)+\dfrac{2^4.3^2.5.7.13}{691}(0,\,11)-\dfrac{2^8.3^4.11}{5.691}\{0\}.$

The above identities will now be established. To start with we shall put down the three particular cases of (18), *viz.*,

$$\Sigma_{1,9}(n)=\frac{5.7.13}{2^2.11.691}\sigma_{11}(n)-\frac{1}{20}n\sigma_9(n)-\frac{2^2.3^2}{5.691}\tau(n), \tag{19}$$

$$\Sigma_{3,7}(n)=\frac{7.13}{2^5.5.691}\sigma_{11}(n)+\frac{3.5}{2^2.691}\tau(n), \tag{20}$$

$$\Sigma_{5,5}(n)=\frac{5.13}{2^2.3^2.7.691}\sigma_{11}(n)-\frac{3}{691}\tau(n). \tag{21}$$

Now

$$\left\{\sum_{n=0}^{\infty}\sigma_5(n)x^n\right\}^2=\sum_{n=0}^{\infty}\Sigma_{5,5}(n)x^n=\frac{1}{2^6.3^4.7^2}+\frac{5.13}{2^2.3^2.7.691}(0,\,11)-\frac{3}{691}\{0\}. \tag{22}$$

Using formula (8) of Table A(1) Part I, we get

$$R^2=1-2^4.3^2.7(0,\,5)+2^6.3^4.7^2(0,\,5)^2. \tag{23}$$

Also putting $r=s=5$ in (5) we have

$$(0,\,5)^2=-\tfrac{1}{4}\{\zeta(-5)\}^2-\zeta(-5)(0,\,5)+\left\{\sum_{n=0}^{\infty}\sigma_5(n)x^n\right\}^2, \tag{24}$$

$$=\frac{1}{2^2.3^2.7}(0,\,5)+\frac{5.13}{2^2.3^2.7.691}(0,\,11)-\frac{3}{691}\{0\},$$

by (22).

Thus (23) and (24) lead to

$$R^2=1+\frac{2^4.3^2.5\;7.13}{691}(0,\,11)-\frac{2^6.3^5.7^2}{691}\{0\},$$

and this is identity (3) of Table A(2).

The second identity in the Table can be similarly derived from identities (6) and (7) of Table A(1) and (20). However we can get it immediately from the formula for R^2 by making use of the wellknown result,

$$Q^3-R^2=1728x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}=1728\sum_{0}^{\infty}\tau(n)x^n. \tag{25}$$

It is now a simple matter to deduce successively the identities (7), (5), (6) and (10) of Table A(2) by making use of the formulae for Q^3 and R^2 and Ramanujan's identities II 5, III 4, 8 and 10 respectively. As for the first formula we need only strengthen Ramanujan's Table III by the addition of the fresh identity

$$248832\Phi_{5,6}(x) = 225P^2Q^2-200P^3R+75P^4Q-120PQR-5P^6+9Q^3+10R^2. \tag{26}$$

This identity can be easily established by the methods outlined by Ramanujan in his paper.

2. In Table B(2) is set down all products of (u,v)'s which can be expressed as a linear function of $\{0\}$ and (u,v)'s. There is nothing which is essentially new in the method of derivation, which is very much the same as that used previously in Part I in connection with Table B(1).

TABLE B(2)

(3.1) $2^4.3.5.7(1,2)(3,4) = \{0\}+2^2.5(4,7)-3.7(5,6),$

(3.2) $2^3.3.5.7(2,3)^2 = -\{0\}+3.5(4,7)-2.7(5,6),$

(3.3) $2^3.3.5.7(0,1)(4,5) = -\{0\}+2.5^2(4,7)-2^2.3.7(5,6)+5.7(4,5);$

(5.1) $2^5.3.7(1,2)(2,5) = \{0\}+7(3,8)-2^3(4,7),$

(5.2) $2^5.3.5.7(2,3)(1,4) = -5\{0\}+3.7(3,8)-2^4(4,7),$

(5.3) $2^5.3.7(0,1)(3,6) = -\{0\}+3.7(3,8)-2^4.3(4,7)+2^2.7(3,6),$

(5.4) $2^5.3.5.7(3,4)(0,3) = 5\{0\}+3.7(3,8)-2^2.3(4,7)-2.7(3,4),$

(5.5) $2^7.3^2.5.7(1,2)^3 = \{0\}+5.7(3,8)-2^3.3.5(4.7)+2^2.3.7(5,6),$

(5.6) $2^8.3^2.5.7(0,1)(1,2)(2,3) = -\{0\}+3.5.7(3,8)-2^3.5.11(4,7)$
$+2^2.5.7(3,6)+2^4.3.7(5,6)-2^2.5.7(4,5),$

(5.7) $2^7.3^2.5.7(0,1)^2(3,4) = \{0\}+3.5.7(3,8)-2^3.3^3.5(4,7)+2^3.5.7(3,6)$
$+2^3.3^2.7(5,6)-2^2.3.5.7(4,5)+2.5.7(3,4);$

(7.1) $2^2.3^3.5(1,4)^2 = -\{0\}+(2,9),$

(7.2) $2^4.3^5.5(0,3)(2,5) = 2^2\{0\}+5(2,9)-3^2(2,5),$

(7.3) $2^4.3^3.7(1,2)(1,6) = 7\{0\}+2^2.5(2,9)-3^3(3,8),$

(7.4) $2^4.3^3(0,1)(2,7) = -\{0\}+2.5(2,9)-3^3(3,8)+2.3^2(2,7),$

(7.5) $2^4.3^3.7(2.3)(0,5) = -7\{0\}+2.5(2,9)-3^2(3,8)+2.3(2,3),$

(7.6) $2^7.3^4.5.7(0,1)(1,2)(1,4) = -5\{0\}+2^2.5.7(2,9)-3^4.7(3,8)+2^2.3^2.7(2,7)$
$+2^4.3^3(4,7)-2^2.3^2.7(3,6),$

(7.7) $2^6.3^4.7(0,1)^2(3,5) = \{0\}+5.7(2,9)-3^3.7(3,8)+2^2.3^3(4,7)$
$-2.3^2.7(2,5)+2.3^2.7(3,4),$

(7.8) $2^7.3^4.5.7(1,2)^2(0,3) = 11\{0\}+2.5.7(2,9)-3^3.7(3,8)+2^2.3^3(4,7)$
$-2.3^2.7(2,5)+2.3^7.7(3,4),$

$$(7.9)\quad 2^{7}.3^{3}.5.7(0,1)(2,3)(0,3) = -2\{0\}+5.7(2,9)-2.3^{2}.7(3,8)+3^{2}.7(2,7)+2^{3}.3^{2}(4,7)$$
$$-2.3.7(3,6)-3^{2}.7(2,5)+2^{2}.3.7(3,4)-3.7(2,3),$$

$$(7.10)\quad 2^{9}.3^{5}.5.7(0,1)^{2}(1,2)^{2} = \{0\}+2.5^{2}.7(2,9)-3^{4}.5.7(3,8)+2^{2}.3^{2}.5.7(2,7)$$
$$+2^{2}.3^{3}.5.13(4,7)-2^{4}.3^{2}.5.7(3,6)+2.3^{2}.5.7(2,5)$$
$$-2^{3}.3^{4}.7(5,6)+2^{2}.3^{3}.5.7(4,5)-2.3^{2}.5.7(3,4),$$

$$(7.11)\quad 2^{9}.3^{4}.5.7(0,1)^{3}(2,3) = -\{0\}+5^{2}.7(2,9)-3^{2}.5^{2}.7(3,8)+3^{3}.5.7(2,7)$$
$$+2^{5}.3^{3}.5(4,7)-2.3^{2}.5.7^{2}(3,6)+3^{3}.5.7(2,5)$$
$$-2^{4}.3^{3}.7(5,6)+2^{2}.3^{3}.5.7(4,5)$$
$$-2^{2}.3^{2}.5.7(3,4)+3.5.7(2,3);$$

$$(9.1)\quad 2^{5}.3^{2}.5^{2}.7(1,4)(0,5) = -2^{2}.3^{2}\{0\}+11(1,10)+5^{2}(1,4),$$

$$(9.2)\quad 2^{4}.3.5^{2}.7(0,3)(1,6) = 2^{3}.3\{0\}+11(1,10)-5.7(1,6),$$

$$(9.3)\quad 2^{7}.3^{2}.5^{2}(0,1)(1,8) = -2^{3}\{0\}+3.11(1,10)-2^{2}.5^{2}(2,9)+3.5^{2}(1,8),$$

$$(9.4)\quad 2^{5}.3^{2}.5^{2}(1,2)(0,7) = 2^{5}\{0\}+3.11(1,10)-2.5^{2}(2,9)-3.5(1,2),$$

$$(9.5)\quad 2^{6}.3^{3}.5.7(0,1)^{2}(1,6) = 2\{0\}+3.11(1,10)-2^{3}.5^{2}(2,9)+2.3.5^{2}(1,8)$$
$$+2.3^{3}.5(3,8)-2^{3}.3^{2}.5(2,7)+3.5.7(1,6),$$

$$(9.6)\quad 2^{7}.3^{3}.5^{3}(0,1)(0,3)(1,4) = -2^{3}\{0\}+3.11(1,10)-2^{2}.5^{2}(2,9)+3.5^{2}(1.8)$$
$$-3.5.7(1,6)+2^{2}.3^{2}.5(2,5)-3.5^{2}(1,4),$$

$$(9.7)\quad 2^{9}.3^{3}.5^{2}.7(0,1)^{3}(1,4) = -2\{0\}+7.11(1,10)-2^{2}.5^{2}.7(2,9)+3.5^{2}.7(1,8)$$
$$+2.3^{3}.5.7(3,8)-2^{4}.3^{2}.5.7(2,7)+3.5.7^{2}(1,6)$$
$$-2^{5}.3^{2}.5(4,7)+2^{3}.3^{2}.5.7(3,6)-2^{2}.3^{3}.5.7(2,5)$$
$$+5^{2}.7(1,4),$$

$$(9.8)\quad 2^{6}.3^{3}.5.7(0,1)(1,2)(0,5) = -\{0\}+11(1,10)-2.5^{2}(2,9)+5^{2}(1,8)+3^{2}.5(3,8)$$
$$-2.3.5(2,7)+5^{2}(1,4)-2.3.5(2,3)+5(1,2).$$

$$(9.9)\quad 2^{8}.3^{3}.5^{3}(1,2)(0,3)^{2} = 2^{5}\{0\}+3.11(1,10)-2.5^{2}(2,9)-2.3.5.7(1,6)$$
$$+2^{2}.3^{2}.5(2,5)+3.5(1,2).$$

$$(9.10)\quad 2^{10}.3^{4}.5^{2}.7(0,1)^{2}(1,2)(0,3) = 2^{2}\{0\}+3.7.11(1,10)-2.5^{3}.7(2,9)+2.3.5^{2}.7(1,8)$$
$$+2^{2}.3^{3}.5.7(3,8)-2^{2}.3^{3}.5.7(2,7)-2^{4}.3^{3}.5(4,7)$$
$$+2^{4}.3^{2}.5.7(3,6)+2^{3}.3^{2}.5.7(2,5)-2.3.5^{2}.7(1,4)$$
$$-2^{3}.3^{2}.5.7(3,4)+2^{2}.3^{2}.5.7(2,3)-3.5.7(1,2);$$

$$(11.1)\quad 2^{7}.3^{2}.7.691(0,5)^{2} = -2^{2}.3^{3}.7\{0\}+5.13(0,11)+691(0,5),$$

$$(11.2)\quad 2^{6}.3.5^{2}.691(0,3)^{3} = 2^{3}.3.5^{2}\{0\}+7.13(0,11)-2.691(0,7)+691(0,3),$$

$$(11.3)\quad 2^{5}.3.5.691(0,3)(0,7) = 2^{4}.3^{2}.5^{2}\{0\}+3.7.13(0,11)-2.691(0,7)-691(0,3),$$

$$(11.4)\quad 2^{3}.3.5.11.691(0,1)(0,9) = -2^{4}.3^{3}.11\{0\}+2.3.5^{2}.7.13(0,11)-2.3.11.691(1,10)$$
$$+5.11.691(0,9)+5.691(0,1),$$

$$\begin{aligned} 2^7.3^2.5^2.691(0,1)^2(0,7) = {} & 2^3.97\{0\}+3.5^2.7.13(0,11)-691[2.3.11(1,10) \\ & -5.11(0,9)-2^2.5^2(2,9)+2.3.5^2(1,8)-2.5^2(0,7) \\ & +5^2(0,3)-2.3.5(1,2)+5(0,1)], \end{aligned} \tag{11.5}$$

$$\begin{aligned} 2^7.3^3.5^2.7.691(0,1)(0,3)(0,5) = {} & -2^5.3^3.11\{0\}+2.3.5^2.7.13(0,11) \\ & -691[2.3.11(1,10)-5.11(0,9)+2^2.5^2(0,7) \\ & -2.3.5.7(1,6)+2.3.5^2(1,4)-2.5^2(0,3)+5(0,1)], \end{aligned} \tag{11.6}$$

$$\begin{aligned} 2^{10}.3^3.5^3.691(0,1)^2(0,3)^2 = {} & 2^3.97\{0\}+3.5^2.7.13(0,11)-691[2.3.11(1,10) \\ & -5.11(0,9)-2^2.5^3(2,9)+2.3.5^3(1,8)+2.5^2(0,7) \\ & -2^2.3.5.7(1,6)+2.3.5.7(0,5)+2^3.3^2.5(2,5) \\ & -2^2.3.5^2(1,4)+5^2(0,3)+2.3.5(1,2)-5(0,1)], \end{aligned} \tag{11.7}$$

$$\begin{aligned} 2^9.3^3.5.7.691(0,1)^3(0,5) = {} & -2^2.101.\{0\}+2.5^2.7.13(0,11)-691[2.3.11(1,10) \\ & -5.11(0,9)-2^3.5^2(2,9)+2^2.3.5^2(1,8)-2^2.5^2(0,7) \\ & +2^3.3^2.5(3,8)-2^3.3^2.5(2,7)+2.3.5.7(1,6) \\ & -2.5.7(0,5)+2.3.5^2(1,4)-2.5^2(0,3) \\ & -2^3.3.5(2,3)+2^2.3.5(1,2)-5(0,1)], \end{aligned} \tag{11.8}$$

$$\begin{aligned} 2^{12}.3^4.5^2.7.691(0,1)^4(0,3) = {} & -2^5.3\{0\}+3.5^2.7^2.13(0,11)-691[2^2.3.7.11(1,10) \\ & -2.5.7.11(0,9)-2^3.3.5^2.7(2,9)+2^2.3^2.5^2.7(1,8) \\ & -2.5^3.7(0,7)+2^3.3^3.5.7(3,8)-2^4.3^3.5.7(2,7) \\ & +2^3.3.5.7^2(1,6)-2^5.3^3.5(4,7)+2^5.3^2.5.7(3,6) \\ & -2^3.3.5^3.7(1,4)+5^3.7(0,3)-2^4.3^2.5.7(3,4) \\ & +2^4.3^2.5.7(2,3)-2^2.3^2.5.7(1,2) \\ & +2.5.7(0,1)], \end{aligned} \tag{11.9}$$

$$\begin{aligned} 2^{14}.3^4.5.7.691(0,1)^6 = {} & -2^6\{0\}+5^2.7^3.13(0,11) \\ & -691[2.3.7.11(1,10)-5.7.11(0,9) \\ & -2^2.5^3.7(2,9)+2.3.5^3.7(1,8) \\ & -2.5^3.7(0,7)+2^3.3^2.5^2.7(3,8) \\ & -2^4.3^2.5^2.7(2,7)+2^2.3.5^2.7^2(1,6) \\ & -2.5^3.7^2(0,5)-2^5.3^3.5^2.(4,7) \\ & +2^5.3^2.5^2.7(3,6)-2^3.3^3.5^2.7(2,5) \\ & +2^2.3.5^3.7(1,4)-5^3.7(0,3) \\ & +2^6.3^2.7(5,6)-2^5.3^3.5.7(4,5) \\ & +2^4.3^2.5^2.7(3,4)-2^4.3.5^2.7(2,3) \\ & +2.3.5^2.7(1,2)-5.7(0,1)]. \end{aligned} \tag{11.10}$$

3. It is obvious that each of the identities in Table B(2) provides us with a congruence relation connecting Ramanujan's function $\tau(n)$ with the divisor functions $\sigma_k(n)$. As an example by equating the coefficientt of x^n on both sides of the identity (5.6) in the Table we get

(27) $$\tau(n) \equiv 105n^3\sigma_5(n)-20(22n^4-7n^3)\sigma_3(n)+28(12n^5-5n^4)\sigma(n) \pmod{2^5.3^2.5.7}.$$

It may be remembered that in our discussion of the method of derivation of congruences from identities of Table B(1) we pointed out that each identity in general provides us with more than one congruence relation. Similary by precisely same methods we shall derive usually more than one congruence relation involving $\tau(n)$ from each formula in Table B(2). By way of further illustration we add two examples showing the process of extraction of these properties, and all congruence relations obtained by the process are set down in Table C(2).

Example 1. We take identity (5.4) in Table B(2) as our starting point:

$$2^5.3.5.7(3, 4)(0, 3) = 5\{0\}+3.7(3, 8)-2^2.3(4, 7)-2.7(3, 4).$$

Now by Theorem A of Part I

$$(3, 4) = (3, 2)+(1, 4)-(1, 2)+g_{3,1}.g_{4,2}J = (2, 3)+(1, 4)-(1, 2)+2^3.3^2J,$$

and

$$(0, 3) = (0, 1)+2.3J.$$

Thus

$$\{(3, 4)-(2, 3)-(1, 4)+(1, 2)\}.\{(0, 3)-(0, 1)\} = 2^4.3^3J.$$

or,

$$\begin{aligned}(3, 4)(0, 3) = {} & (2, 3)(0, 3)+(0, 3)(1, 4)-(1, 2)(0, 3)+(0, 1)(3, 4)\\ & -(0, 1)(2, 3)-(0, 1)(1,4)+(0, 1)(1, 2)+2^4.3^3J,\end{aligned}$$

which of course gives us

$$\begin{aligned}2^5.3.5.7(3, 4)(0, 3) = {} & 14\times 2^4.3.5(2, 3)(0, 3)+14\times 2^4.3.5(0, 3)(1, 4)\\ & -14\times 2^4.3.5(1, 2)(0, 3)+140\times 2^3.3(0, 1)(3, 4)-140.\times 2^3.3(0, 1)(2, 3)\\ & -28\times 2^3.3.5(0, 1)(1, 4)+140\times 2^3.3(0, 1)(1, 2)+2^9.3^4.5.7J.\end{aligned}$$

Now by making use of results (5.4) of Table B(2) and (5.8), (7.5), (5.4), (3.6), (3.4), (6.3) and (3.2) of Table B(1) we get

$$\begin{aligned}5\{0\}+3.7(3, 8)-2^2.3(4, 7)-2\cdot 7(3, 4) = {} & 14[3(2, 7)-2(3, 6)-(2, 3)]\\ & +14[1, 8)-(1, 4)]-14[7(1, 6)-2.3(2, 5)-(1, 2)]\\ & +140[2(3, 6)-3(4, 5)+(3, 4)]-140[3(2, 5)-2^2(3, 4)\\ & +(2, 3)]-28[7(1.6)-2^2.3(2, 5)+5(1, 4)]\\ & +140[5(1, 4)-2.3(2, 3)+(1, 2)]+2^9.3^4.5.7J,\end{aligned}$$

or,

$$\begin{aligned}5\{0\} = {} & 14(1, 8)+[-21(3, 8)+42(2, 7)-294(1, 6)]+[12(4, 7)+252(3, 6)+546(1, 4)]\\ & +[-420(4, 5)+714(3, 4)-994(2, 3)+154(1, 2)]+2^7.3^4.5.7J.\end{aligned}$$

Hence equating the coefficients of x^n on both sides we get

$$(28)\quad 5\tau(n) \equiv 14n\sigma_7(n)-21(n^3-2n^2+14n)\sigma_5(n)$$
$$+6(2n^4+42n^3+91n)\sigma_3(n)-14(30n^4-51n^3+71n^2-11n)\sigma(n) \pmod{2^9.3^4.5.7}.$$

This is formula (7.35) of Table C(2).

Example 2. Here we start from identity (11.2), *viz.*,

$$2^8.3.5^2.691(0,\ 3)^3 = 2^3.3.5^2\{0\}+7.13(0,\ 11)-2.691(0,\ 7)+691(0,\ 3).$$

Now from Theorem A,

$$(0,\ 3) = (0,\ 1)+2.3.J.$$

Therefore

$$\{(0,\ 3)-(0,\ 1)\}^3 = 2^3.3^3J,$$

or,

$$(0,\ 3)^3 = 3(0,\ 1)(0,\ 3)^2-3(0,\ 1)^2(0,\ 3)+(0,\ 1)^3+2^3.3^3J.$$

Hence

$$2^8.3.5^2.691(0,\ 3)^3 \doteq 691\times 2^3.3^2.5^2(0,\ 1)(0,\ 3)^2-20.691\times 2^6.3^2.5(0,\ 1)^2(0,\ 3)$$
$$+100.691\times 2^6.3(0,\ 1)^3+2^{11}.3^4.5^2.691J.$$

Now making use of identities (11.2) of Table B(2) and (9.3), (7.3) and (5.2) of Table B(1) we get

$$2^3.3.5^2\{0\}+7.13(0,\ 11)-2.691(0,\ 7)+691(0,\ 3) = 691[11(0,\ 9)-2.3.5(1,\ 8)$$
$$+2^2.5(0,\ 7)-2.3.7(0,\ 5)+2^2.3.5(1,\ 4)-2^3.5(0,\ 3)+(0,\ 1)]-20.691[2.5(0,\ 7)$$
$$-2.3.7(1,\ 6)+3.7(0,\ 5)+2^2.3^2(2,\ 5)-2.3.5(1,\ 4)+2.3(1,\ 2)-(0,\ 1)]$$
$$+100.691[7(0,\ 5)-2.3.5(1,\ 4)+2.5(0,\ 3)+2^3.3(2,\ 3)$$
$$-2^2.3(1,\ 2)+(0,\ 1)]+2^{11}.3^4.5^2.691J.$$

or,

$$2^3.3.5^2\{0\} = -91(0,\ 11)+691[11(0,\ 9)+\{-30(1,\ 8)-178(0,\ 7)\}$$
$$+\{840(1,\ 6)+238(0,\ 5)\}+\{-720(2,\ 5)-2340(1,\ 4)+979(0,\ 3)\}$$
$$+\{2400(2,\ 3)-1320(1,\ 2)+121(0,\ 1)\}]+2^{11}.3^4.5^2.691J.$$

By equating the coefficients of x^n on both sides we get

$$(29)\quad 2^3.3.5^2\tau(n) \equiv -91\sigma_{11}(n)+691[11\sigma_9(n)-2(15n+89)\sigma_7(n)$$
$$+14(60n+17)\sigma_5(n)-(720n^2+2340n-979)\sigma_3(n)$$
$$+(2400n^2-1320n+121)\sigma(n)] \pmod{2^{11}.3^4.5^2.691}.$$

and this is formula (11.8) in Table C(2).

We give below the rather long Table C(2). The figures noted within square brakets against each congruence relation refer to the corresponding identity in Table B(2) from which it is derived. (The congruences marked [13.1], [13.2], etc. are deduced from the corresponding identities in Table B(1)).

TABLE C(2)

(3.1) $\tau(n) \equiv -20n^4\sigma_3(n) + 21n^5\sigma(n)$ (mod $2^3.3.5.7$), [3.1]
(3.2) $\tau(n) \equiv 15n^4\sigma_3(n) - 14n^5\sigma(n)$ (mod $2^3.3.5.7$), [3.2]
(3.3) $\tau(n) \equiv 50n^4\sigma_3(n) - 7(12n^5 - 5n^4)\sigma(n)$ (mod $2^3.3.5,7$), [3.3]
(3.4) $\tau(n) \equiv 5(3n^4 - 7n^3)\sigma_3(n) - 7(2n^5 - 5n^4)\sigma(n)$ (mod $2^4.3^2.5.7$), [3.2]
(3.5) $\tau(n) \equiv -5(4n^4 - 7n^3)\sigma_3(n) + 7(3n^5 - 5n^4)\sigma(n)$ (mod $2^5.3^2.5.7$), [3.1]
(3.6) $\tau(n) \equiv 10(5n^4 - 7n^3)\sigma_3(n) - 7(12n^5 - 20n^4 + 5n^3)\sigma(n)$ (mod $2^6.3^2.5.7$), [3.3]
(3.7) $\tau(n) \equiv 5(3n^4 - 14n^3 + 14n^2)\sigma_5(n) - 14(n^5 - 5n^4 + 5n^3)\sigma(n)$ (mod $2^5.3^3.5.7$); [3.2]

(5.1) $\tau(n) \equiv -7n^3\sigma_5(n) + 8n^4\sigma_3(n)$ (mod $2^5.3.7$), [5.1]
(5.2) $\tau(n) \equiv 21n^3\sigma_5(n) - 4(12n^4 - 7n^3)\sigma_3(n)$ (mod $2^5.3.7$), [5.3]
(5.3) $\tau(n) \equiv -7n^3\sigma_5(n) + 4(2n^4 + 7n^3)\sigma_3(n) - 28n^4\sigma(n)$ (mod $2^9.3^2.7$), [5.1]
(5.4) $\tau(n) \equiv 21n^3\sigma_5(n) - 4(12n^4 + 7n^3)\sigma_3(n) + 28(3n^4 - n^3)\sigma(n)$ (mod $2^8.3^2.7$), [5.3]
(5.5) $5\tau(n) \equiv 21n^3\sigma_5(n) - 16n^4\sigma_3(n)$ (mod $2^5.3.5.7$), [5 2]
(5.6) $5\tau(n) \equiv -21n^3\sigma_5(n) + 12n^4\sigma_3(n) + 14n^3\sigma(n)$ (mod $2^5.3.5.7$), [5.4]
(5.7) $\tau(n) \equiv 14n^2\sigma_5(n) - 2(10n^4 + 7n^3)\sigma_3(n) + 21n^5\sigma(n)$ (mod $2^4.3^2.5.7$), [3.1]
(5.8) $\tau(n) \equiv -35n^2\sigma_5(n) + 5(10n^4 + 14n^3 - 7n^2)\sigma_3(n)$
$-7(12n^5 - 5n^4)\sigma(n)$ (mod $2^5.3^2.5.7$), [3.3]
(5.9) $\tau(n) \equiv -7n^3\sigma_5(n) + 8(n^4 + 7n^2)\sigma_3(n) - 56n^3\sigma(n)$ (mod $2^6.3^2.5\ 7$), [5.1]
(5.10) $5\tau(n) \equiv 7(3n^3 - 8n^2)\sigma_5(n) - 8(2n^4 - 7n^3)\sigma_3(n)$ (mod $2^6.3^2.5.7$), [5.2]
(5.11) $5\tau(n) \equiv -21n^3\sigma_5(n) + 4(3n^4 + 70n^3)\sigma_3(n)$
$-14(30n^4 - 11n^3)\sigma(n)$ (mod $2^6.3^2.5.7$), [5.4]
(5.12) $5\tau(n) \equiv -21(n^3 - 2n^2)\sigma_5(n) + 4(3n^4 - 7n^3)\sigma_3(n)$
$+14(n^3 - n^2)\sigma(n)$ (mod $2^7.3^2.5\ 7$); [5.4]
(5.13) $\tau(n) \equiv 21n^3\sigma_5(n) - 4(12n^4 - 7n^3 + 21n^2)\sigma_3(n)$
$+28(4n^3 - n^2)\sigma(n)$ (mod $2^7.3^2.5.7$), [5.3]
(5.14) $5\tau(n) \equiv 21n^3\sigma_5(n) - 4(4n^4 + 35n^3)\sigma_3(n) + 140n^4\sigma(n)$ (mod $2^7.3^2.5.7$), [5.2]
(5.15) $\tau(n) \equiv -35n^3\sigma_5(n) + 120n^4\sigma_3(n) - 84n^5\sigma(n)$ (mod $2^7.3^2.5.7$), [5.5]
(5.16) $\tau(n) \equiv -105n^3\sigma_5(n) + 20(27n^4 - 14n^3)\sigma_3(n)$
$-14(36n^5 - 30n^4 + 5n^3)\sigma(n)$ (mod $2^7.3^2.5.7$), [5.7]
(5.17) $\tau(n) \equiv 105n^3\sigma_5(n) - 20(22n^4 - 7n^3)\sigma_3(n)$
$+28(12n^5 - 5n^4)\sigma(n)$ (mod $2^8.3^2.5.7$), [5.6]
(5.18) $\tau(n) \equiv 14n^2\sigma_5(n) - (20n^4 - 21n^3 + 70n^2)\sigma_3(n)$
$+7(3n^5 - 5n^4 + 10n^3)\sigma(n)$ (mod $2^6.3^3.5.7$), [3.1]
(5.19) $\tau(n) \equiv -35n^2\sigma_5(n) + 10(5n^4 + 7n^2)\sigma_3(n)$
$-7(12n^5 - 20n^4 + 25n^3 - 5n^2)\sigma(n)$ (mod $2^8.3^3,5.7$), [3.3]

(5.20) $5\tau(n) \equiv 7(3n^3-8n^2)\sigma_5(n)-4(4n^4+21n^3-70n^2)\sigma_3(n)$
$\qquad +140(n^4-2n^3)\sigma(n) \pmod{2^7.3^3.5.7}$, [5.2]

(5.21) $5\tau(n) \equiv -21(n^3-2n^2)\sigma_5(n)+12(n^4+21n^3-35n^2)\sigma_3(n)$
$\qquad -14(30n^4-51n^3+11n^2)\sigma(n) \pmod{2^8.3^3.5.7}$, [5.4]

(5.22) $\tau(n) \equiv 35(3n^3-8n^2)\sigma_5(n)-20(22n^4-63n^3+14n^2)\sigma_3(n)$
$\qquad +28(12n^5-35n^4+10n^3)\sigma(n) \pmod{2^9.3^3.5.7}$, [5.6]

(5.23) $\tau(n) \equiv -105(n^3-2n^2)\sigma_5(n)+60(9n^4-21n^3+7n^2)\sigma_3(n)$
$\qquad -14(36n^5-90n^4+45n^3-5n^2)\sigma(n) \pmod{2^9.3^3.5.7}$, [5.7]

(5.24) $\tau(n) \equiv -49n\sigma_5(n)+(50n^4+84n^2-35n)\sigma_3(n)$
$\qquad -7(12n^5-5n^4)\sigma(n) \pmod{2^4.3^2.5^2.7}$, [3.3]

(5.25) $\tau(n) \equiv -7(5n^2+7n)\sigma_5(n)+(50n^4+70n^3+49n^2$
$\qquad +140n)\sigma_3(n)-7(12n^5-5n^4+30n^2-5n)\sigma(n) \pmod{2^6.3^3.5^2.7}$; [3.3]

(7.1) $\tau(n) \equiv 10n^2\sigma_7(n)-9(3n^3-2n^2)\sigma_5(n) \pmod{2^4.3^3}$, [7.4]

(7.2) $\tau(n) \equiv 10n^2\sigma_7(n)-27n^3\sigma_5(n)+18(2n^3-n^2)\sigma_3(n) \pmod{2^7.3^4}$, [7.4]

(7.3) $\tau(n) \equiv n^2\sigma_7(n) \pmod{2^2.3^3.5}$, [7.1]

(7.4) $4\tau(n) \equiv -5n^2\sigma_7(n)+9n^2\sigma_3(n) \pmod{2^4.3^3.5}$; [7.2]

(7.5) $\tau(n) \equiv n^2\sigma_7(n)-9n^2\sigma_5(n)+9n^3\sigma_3(n) \pmod{2^4.3^4.5}$, [7.1]

(7.6) $4\tau(n) \equiv -5n^2\sigma_7(n)+90n^2\sigma_5(n)-9(20n^3-11n^2)\sigma_3(n) \pmod{2^5.3^4.5}$, [7.2]

(7.7) $4\tau(n) \equiv -5n^2\sigma_7(n)+27n^2\sigma_5(n)-9(2n^3-n^2)\sigma_3(n)-9n^2\sigma(n) \pmod{2^7.3^4.5}$, [7.2]

(7.8) $\tau(n) \equiv 10n^2\sigma_7(n)-9(3n^3-2n^2)\sigma_5(n)$
$\qquad -54n^2\sigma_3(n)+18(4n^3-n^2)\sigma(n) \pmod{2^7.3^4.5}$, [7.4]

(7.9) $\tau(n) \equiv n^2\sigma_7(n)-18n^2\sigma_5(n)+9(2n^3+5n^2)\sigma_3(n)-45n^3\sigma(n) \pmod{2^6.3^5.5}$, [7.1]

(7.10) $4\tau(n) \equiv -5n^2\sigma_7(n)+117n^2\sigma_5(n)-9(22n^3+19n^2)\sigma_3(n)$
$\qquad +9(40n^3-11n^2)\sigma(n) \pmod{2^8.3^5.5}$, [7.2]

(7.11) $4\tau(n) \equiv -5n^2\sigma_7(n)+63n\sigma_5(n)-45n^2\sigma_3(n)-9n\sigma(n) \pmod{2^5.3^4.5^2}$, [7.2]

(7.12) $4\tau(n) \equiv -5n^2\sigma_7(n)+9(10n^2+7n)\sigma_5(n)$
$\qquad -9(20n^3-5n^2+50n)\sigma_3(n)+9(60n^2-11n)\sigma(n) \pmod{2^6.3^5.5^2}$, [7.2]

(7.13) $\tau(n) \equiv -20n\sigma_7(n)+(21n^3+48n^2-28n)\sigma_5(n)$
$\qquad -4(12n^4-7n^3)\sigma_3(n) \pmod{2^6.3^2.7}$, [5.3]

(7.14) $7\tau(n) \equiv -20n^2\sigma_7(n)+27n^3\sigma_5(n) \pmod{2^4.3^3.7}$, [7.3]

(7.15) $7\tau(n) \equiv 10n^2\sigma_7(n)-9n^3\sigma_5(n)+6n^2\sigma(n) \pmod{2^4.3^3.7}$, [7.5]

(7.16) $\tau(n) \equiv 10n^2\sigma_7(n)-9(3n^3-2n^2)\sigma_5(n)-90n\sigma_3(n)$
$\qquad +18(6n^2-n)\sigma(n) \pmod{2^5.3^4.7}$, [7.4]

(7.17) $7\tau(n) \equiv 10(n^2-3n)\sigma_7(n)-9(n^3-4n^2)\sigma_5(n)+6(n^2-n)\sigma(n) \pmod{2^5.3^4.7}$, [7.5]

$$\begin{aligned}(7.18)\qquad \tau(n) \equiv {} & -35n^2\sigma_7(n)+63(3n^3-2n^2)\sigma_5(n)\\ & -9(24n^4-28n^3+7n^2)\sigma_3(n) \quad (\text{mod } 2^6.3^4.7), \quad [7.7]\end{aligned}$$

$$\begin{aligned}(7.19)\qquad \tau(n) \equiv {} & -35n^2\sigma_7(n)+63(3n^3+n^2)\sigma_5(n)-9(24n^4\\ & +70n^3-35n^2)\sigma_3(n)+63(12n^4-8n^3+n^2)\sigma(n) \quad (\text{mod } 2^9.3^5.7), \quad [7.7]\end{aligned}$$

$$(7.20)\qquad 5\tau(n) \equiv 14n\sigma_7(n)-21n^3\sigma_5(n)+2(6n^4-7n)\sigma_3(n)+14n^3\sigma(n) \quad (\text{mod } 2^6.3^2.5.7), \quad [5.4]$$

$$\begin{aligned}(7.21)\qquad 5\tau(n) \equiv {} & 14n\sigma_7(n)-7(3n^3+28n)\sigma_5(n)+2(6n^4+140n^3\\ & +168n^2-77n)\sigma_3(n)-14(30n^4-11n^3)\sigma(n) \quad (\text{mod } 2^7.3^3.5.7), \quad [5.4]\end{aligned}$$

$$\begin{aligned}(7.22)\qquad 2\tau(n) \equiv {} & 35n^2\sigma_7(n)-63(2n^3-n^2)\sigma_5(n)\\ & +3(24n^4-14n^3-21n^2)\sigma_3(n)+21(4n^3-n^2)\sigma(n) \quad (\text{mod } 2^7.3^3.5.7), \quad [7.9]\end{aligned}$$

$$\begin{aligned}(7.23)\qquad 5\tau(n) \equiv {} & 14n\sigma_7(n)-7(3n^3-6n^2+14n)\sigma_5(n)\\ & +2(6n^4-14n^3+42n^2-7n)\sigma_3(n)\\ & +14(n^3-n^2+n)\sigma(n) \quad (\text{mod } 2^5.3^3.5.7), \quad [5.4]\end{aligned}$$

$$\begin{aligned}(7.24)\qquad \tau(n) \equiv {} & -20n\sigma_7(n)+(21n^3+48n^2-28n)\sigma_5(n)\\ & -4(12n^4-7n^3+21n^2-35n)\sigma_3(n)\\ & +28(4n^3-7n^2+n)\sigma(n) \quad (\text{mod } 2^3.3^3.5.7), \quad [5.3]\end{aligned}$$

$$\begin{aligned}(7.25)\qquad \tau(n) \equiv {} & 70n\sigma_7(n)-7(15n^3+48n^2-28n)\sigma_5(n)\\ & +2(270n^4+28n^3-168n^2+35n)\sigma_3(n)\\ & -14(36n^5-30n^4+5n^3)\sigma(n) \quad (\text{mod } 2^4.3^3.5.7), \quad [5.7]\end{aligned}$$

$$\begin{aligned}(7.26)\qquad 5\tau(n) \equiv {} & -100n\sigma_7(n)+(105n^3+240n^2+56n)\sigma_5(n)\\ & -4(60n^4+35n^3+84n^2-35n)\sigma_3(n)\\ & +140(3n^4-n^3)\sigma(n) \quad (\text{mod } 2^9.3^3.5.7), \quad [5.3]\end{aligned}$$

$$\begin{aligned}(7.27)\qquad 7\tau(n) \equiv {} & 10n^2\sigma_7(n)-9n^3\sigma_5(n)-378n^2\sigma_3(n)\\ & +24(21n^3-5n^2)\sigma(n) \quad (\text{mod } 2^5.3^4.5.7), \quad [7\ 5]\end{aligned}$$

$$(7.28)\qquad 7\tau(n) \equiv -20n^2\sigma_7(n)+27n^3\sigma_3(n)+252n^2\sigma_3(n)-252n^3\sigma(n) \quad (\text{mod } 2^6.3^4.5.7), \quad [7.3]$$

$$(7.29)\qquad 35\tau(n) \equiv -100n^2\sigma_7(n)+9(15n^3+28n^2)\sigma_5(n)-252n^3\sigma_3(n) \quad (\text{mod } 2^7.3^4.5.7), \quad [7.3]$$

$$\begin{aligned}(7.30)\qquad 35\tau(n) \equiv {} & 50n^2\sigma_7(n)-9(5n^3+21n^2)\sigma_5(n)\\ & +126n^3\sigma_3(n)+93n^2\sigma(n) \quad (\text{mod } 2^7.3^4.5.7), \quad [7.5]\end{aligned}$$

$$\begin{aligned}(7.31)\qquad 5\tau(n) \equiv {} & 140n^2\sigma_7(n)-63(9n^3-4n^2)\sigma_5(n)\\ & +36(12n^4-7n^3)\sigma_3(n) \quad (\text{mod } 2^7.3^4.5.7), \quad [7.6]\end{aligned}$$

$$\begin{aligned}(7.32)\qquad 11\tau(n) \equiv {} & -70n^2\sigma_7(n)+189n^3\sigma_5(n)\\ & -18(6n^4-7n^2)\sigma_3(n)-126n^3\sigma(n) \quad (\text{mod } 2^7.3^4.5.7), \quad [7.8]\end{aligned}$$

$$\begin{aligned}(7.33)\qquad 2\tau(n) \equiv {} & 35n^2\sigma_7(n)-63(2n^3-11n^2)\sigma_5(n)\\ & +3(24n^4-994n^3+399n^2)\sigma_3(n)\\ & +21(120n^4-76n^3+9n^2)\sigma(n) \quad (\text{mod } 2^8,3^4.5.7), \quad [7.9]\end{aligned}$$

(7.34) $$\begin{aligned}2\tau(n) \equiv {} & 35(n^2+3n)\sigma_7(n)-21(6n^3+15n^2-7n)\sigma_5(n)\\ & +3(24n^4+70n^3-63n^2-35n)\sigma_3(n)\\ & +21(4n^3+5n^2-n)\sigma(n) \pmod{2^8.3^4.5.7}, \quad [7.9]\end{aligned}$$

(7.35) $$\begin{aligned}5\tau(n) \equiv {} & 14n\sigma_7(n)-21(n^3-2n^2+14n)\sigma_5(n)\\ & +6(2n^4+42n^3+91n)\sigma_3(n)\\ & -14(30n^4-51n^3+71n^2-11n)\sigma(n) \pmod{2^9.3^4.5.7}, \quad [5.4]\end{aligned}$$

(7.36) $$\begin{aligned}\tau(n) \equiv {} & 175n^2\sigma_7(n)-315(5n^3-3n^2)\sigma_5(n)\\ & +45(96n^4-98n^3+21n^2)\sigma_3(n)\\ & -21(144n^5-180n^4+60n^3-5n^2)\sigma(n) \pmod{2^9.3^4.5.7}, \quad [7.11]\end{aligned}$$

(7.37) $$\begin{aligned}\tau(n) \equiv {} & 70n\sigma_7(n)-21(5n^3+6n^2+14n)\sigma_5(n)\\ & +6(90n^4-154n^3+364n^2-105n)\sigma_3(n)\\ & -14(36n^5-90n^4+165n^3-65n^2+5n)\sigma(n) \pmod{2^{10}.3^4.5.7}, \quad [5.7]\end{aligned}$$

(7.38) $$\begin{aligned}7\tau(n) \equiv {} & 10(n^2-3n)\sigma_7(n)-9(n^3-4n^2)\sigma_5(n)\\ & -126(3n^2-5n)\sigma_3(n)+12(42n^3-73n^2+10n)\sigma(n) \pmod{2^6.3^3.5.7}, \quad [7.5]\end{aligned}$$

(7.39) $$\begin{aligned}\tau(n) \equiv {} & -35n^2\sigma_7(n)+63(3n^3-2n^2+7n)\sigma_5(n)\\ & -9(24n^4-28n^3+217n^2-70n)\sigma_3(n)\\ & +63(24n^3-12n^2+n)\sigma(n) \pmod{2^7.3^5.5.7}, \quad [7.7]\end{aligned}$$

(7.40) $$\begin{aligned}11\tau(n) \equiv {} & -70n^2\sigma_7(n)+63(3n^3+20n^2)\sigma_5(n)\\ & -2(54n^4+2520n^3-693n^2)\sigma_3(n)\\ & +126(30n^4-11n^3)\sigma(n) \pmod{2^8.3^5.5.7}, \quad [7.8]\end{aligned}$$

(7.41) $$\begin{aligned}35\tau(n) \equiv {} & 50(n^2-3n)\sigma_7(n)-9(5n^3+n^2-49n)\sigma_5(n)\\ & +126(n^3-3n^2)\sigma_3(n)+93(n^2-n)\sigma(n) \pmod{2^8.3^5.5.7}, \quad [7.5]\end{aligned}$$

(7.42) $$\begin{aligned}5\tau(n) \equiv {} & 140n^2\sigma_7(n)-63(9n^3+16n^2)\sigma_5(n)\\ & +36(12n^4+133n^3-35n^2)\sigma_3(n)\\ & -1260(3n^4-n^3)\sigma(n) \pmod{2^9.3^5.5.7}, \quad [7.6]\end{aligned}$$

(7.43) $$\begin{aligned}\tau(n) \equiv {} & -350n^2\sigma_7(n)+315(9n^3-4n^2)\sigma_5(n)\\ & -90(78n^4-56n^3+7n^2)\sigma_3(n)\\ & +126(36n^5-30n^4+5n^3)\sigma(n) \pmod{2^9.3^5.5.7}, \quad [7.10]\end{aligned}$$

(7.44) $$\begin{aligned}2\tau(n) \equiv {} & 35(n^2-3n)\sigma_7(n)-63(2n^3+3n^2-21n)\sigma_5(n)\\ & +3(24n^4+882n^3-2499n^2+735n)\sigma_3(n)\\ & -21(120n^4-324n^3+137n^2-11n)\sigma(n) \pmod{2^9.3^5.5.7}, \quad [7.9]\end{aligned}$$

(7.45) $$\begin{aligned}\tau(n) \equiv {} & 175(n^2-3n)\sigma_7(n)-315(5n^3-17n^2+7n)\sigma_5(n)\\ & +45(96n^4-350n^3+231n^2-35n)\sigma_3(n)\\ & -21(144n^5-540n^4+420n^3-95n^2+5n)\sigma(n) \pmod{2^{10}.3^5.5.7}; \quad [7.11]\end{aligned}$$

$$(9.1)\qquad 8\tau(n) \equiv 33n\sigma_9(n) - 25(4n^2-3n)\sigma_7(n) \pmod{2^3.3^2.5^2}; \qquad [9.3]$$

$$(9.2)\qquad 32\tau(n) \equiv -33n\sigma_9(n) + 50n^2\sigma_7(n) + 15n\sigma(n) \pmod{2^5.3^2.5^2}, \qquad [9.4]$$

$$\begin{aligned}(9.3)\qquad 8\tau(n) \equiv\ & 33n\sigma_9(n) - 25(4n^2-3n)\sigma_7(n)\\ & -105n\sigma_5(n) + 15(12n^2-5n)\sigma_3(n) \pmod{2^7.3^3.5^3}, \qquad [9.3,\ 9.6]\end{aligned}$$

$$\begin{aligned}(9.4)\qquad 32\tau(n) \equiv\ & -33n\sigma_9(n) + 50n^2\sigma_7(n)\\ & +210n\sigma_5(n) - 180n^2\sigma_3(n) - 15n\sigma(n) \pmod{2^8.3^3.5^3}, \qquad [9.4,\ 9.9]\end{aligned}$$

$$\begin{aligned}(9.5)\qquad 8\tau(n) \equiv\ & 33n\sigma_9(n) - 25(4n^2+27n)\sigma_7(n)\\ & +45(80n^2-49n)\sigma_5(n)\\ & -15(240n^3-252n^2+55n)\sigma_3(n) \pmod{2^8.3^4.5^3}, \qquad [9.6]\end{aligned}$$

$$\begin{aligned}(9.6)\qquad 32\tau(n) \equiv\ & -33n\sigma_9(n) + 50(n^2+15n)\sigma_7(n)\\ & -90(30n^2-14n)\sigma_5(n)\\ & +30(60n^3-36n^2-25n)\sigma_3(n)\\ & +15(60n^2-11n)\sigma(n) \pmod{2^9.3^4.5^3}, \qquad [9.9]\end{aligned}$$

$$\begin{aligned}(9.7)\qquad 8\tau(n) \equiv\ & 33n\sigma_9(n) - 100(n^2+3n)\sigma_7(n)\\ & +45(30n^2-14n)\sigma_5(n) - 30(80n^3\\ & -21n^2-10n)\sigma_3(n) - 75(6n^2-n)\sigma(n) \pmod{2^9.3^4.5^3}, \qquad [9.6]\end{aligned}$$

$$\begin{aligned}(9.8)\qquad 32\tau(n) \equiv\ & -33n\sigma_9(n) + 50(n^2+30n)\sigma_7(n)\\ & -90(60n^2+91n)\sigma_5(n)\\ & +30(120n^3+1434n^2-550n)\sigma_3(n)\\ & -15(2400n^3-1320n^2+121n)\sigma(n) \pmod{2^{10}.3^5.5^3}, \qquad [9.9]\end{aligned}$$

$$\begin{aligned}(9.9)\qquad 8\tau(n) \equiv\ & 33n\sigma_9(n) - 50(2n^2+21n)\sigma_7(n)\\ & +90(55n^2+28n)\sigma_5(n)\\ & -30(150n^3+609n^2-235n)\sigma_3(n)\\ & +75(240n^3-126n^2+11n)\sigma(n) \pmod{2^{10}.3^5.5^3}, \qquad [9.6]\end{aligned}$$

$$\begin{aligned}(9.10)\qquad 2\tau(n) \equiv\ & -33n\sigma_9(n) + 50(4n^2-3n)\sigma_7(n)\\ & -15(18n^3-24n^2+7n)\sigma_3(n) \pmod{2^6.3^3.5.7}, \qquad [9.5]\end{aligned}$$

$$\begin{aligned}(9.11)\qquad \tau(n) \equiv\ & 11n\sigma_9(n) - 25(2n^2-n)\sigma_7(n)\\ & +15(8n^3-2n^2)\sigma_5(n) + 25n\sigma_3(n)\\ & -5(6n^2-n)\sigma(n) \pmod{2^6.3^5.5.7}, \qquad [9.8]\end{aligned}$$

$$\begin{aligned}(9.12)\qquad 2\tau(n) \equiv\ & -33n\sigma_9(n) + 5(40n^2-9n)\sigma_7(n)\\ & -9(30n^3+16n^2-21n)\sigma_5(n)\\ & +21(24n^3-24n^2+5n)\sigma_3(n) \pmod{2^9.3^4.5.7}, \qquad [9.5]\end{aligned}$$

$$\begin{aligned}(9.13)\qquad 2\tau(n) \equiv\ & 22n\sigma_9(n) - 5(20n^2+11n)\sigma_7(n)\\ & +3(30n^3+106n^2-49n)\sigma_3(n) - (252n^3\\ & -126n^2-155n)\sigma_5(n) - 31(6n^2-n)\sigma(n) \pmod{2^{10}.3^4.5.7}, \qquad [9.8]\end{aligned}$$

(9.14) $24\tau(n) \equiv -11n\sigma_9(n) + 35n\sigma_5(n) \pmod{2^4.3.5^2.7}$, [9.2]

(9.15) $36\tau(n) \equiv 11n\sigma_9(n) + 25n\sigma_3(n) \pmod{2^3.3^2.5^2.7}$, [9.1]

(9.16) $24\tau(n) \equiv -11n\sigma_9(n) + 250n\sigma_7(n)$
$-5(120n^2 - 77n)\sigma_5(n) \pmod{2^5.3^2.5^2.7}$, [9.2]

(9.17) $24\tau(n) \equiv -11n\sigma_9(n) + 35n\sigma_7(n) + 35n\sigma_5(n)$
$-35n\sigma_3(n) \pmod{2^7.3^2.5^2.7}$, [9.2]

(9.18) $36\tau(n) \equiv 11n\sigma_9(n) - 125n\sigma_7(n)$
$+150n^2\sigma_3(n) + 25n\sigma_3(n) - 25n\sigma(n) \pmod{2^5.3^3.5^2.7}$, [9.1]

(9.19) $56\tau(n) \equiv 231n\sigma_9(n) - 50(14n^2 - 3n)\sigma_7(n)$
$+75(12n^2 - 7n)\sigma_5(n) \pmod{2^6.3^3.5^2.7}$, [9.3]

(9.20) $32\tau(n) \equiv -33n\sigma_9(n) + 50n^2\sigma_7(n) + 1500n\sigma_3(n)$
$-45(40n^2 - 7n)\sigma(n) \pmod{2^6.3^3.5^2.7}$, [9.4]

(9.21) $72\tau(n) \equiv 22n\sigma_9(n) - 105n\sigma_7(n) + 155n\sigma_3(n) \pmod{2^7.3^3.5^2.7}$, [9.1]

(9.22) $24\tau(n) \equiv -11n\sigma_9(n) + 285n\sigma_7(n)$
$-15(40n^2 + 7n)\sigma_5(n) + 35(24n^2 - 11n)\sigma_3(n) \pmod{2^8.3^3.5^2.7}$, [9.2]

(9.23) $224\tau(n) \equiv -231n\sigma_9(n) + 50(7n^2 + 10n)\sigma_7(n)$
$-600n^2\sigma_5(n) + 205n\sigma(n) \pmod{2^8.3^3.5^2.7}$, [9.4]

(9.24) $2\tau(n) \equiv 77n\sigma_9(n) - 175(4n^2 - 3n)\sigma_7(n)$
$+105(18n^3 - 24n^2 + 7n)\sigma_5(n)$
$-5(288n^4 - 504n^3 + 252n^2 - 35n)\sigma_3(n) \pmod{2^9.3^3.5^2.7}$, [9.7]

(9.25) $8\tau(n) \equiv 33n\sigma_9(n) - 25(4n^2 - 3n)\sigma_7(n)$
$-375n\sigma_3(n) + 75(6n^2 - n)\sigma(n) \pmod{2^5.3^4.5^2.7}$, [9.3]

(9.26) $\tau(n) \equiv 11n\sigma_9(n) - 25(2n^2 - n)\sigma_7(n) + 15(3n^3$
$-2n^2 - 49n)\sigma_5(n) + 25(126n^2 - 41n)\sigma_3(n)$
$-5(504n^3 - 246n^2 + 20n)\sigma(n) \pmod{2^7.3^4.5^2.7}$, [9.8]

(9.27) $2\tau(n) \equiv -33n\sigma_9(n) + 50(4n^2 - 3n)\sigma_7(n) - 90(3n^3$
$-4n^2 - 7n)\sigma_5(n) - 1050(3n^2 - n)\sigma_3(n)$
$+105(24n^3 - 12n^2 + n)\sigma(n) \pmod{2^8.3^4.5^2.7}$, [9.5]

(9.28) $72\tau(n) \equiv 22n\sigma_9(n) - 355n\sigma_7(n)$
$+15(20n^2 + 49n)\sigma_5(n)$
$-5(126n^2 - 31n)\sigma_3(n) - 155n\sigma(n) \pmod{2^9.3^4.5^2.7}$, [9.1]

(9.29) $4\tau(n) \equiv -231n\sigma_9(n) + 350(5n^2 - 3n)\sigma_7(n)$
$-3780(n^3 - n^2)\sigma_5(n)$
$+30(72n^4 - 84n^3 - 84n^2 + 35n)\sigma_3(n)$
$+105(24n^3 - 12n^2 + n)\sigma(n) \pmod{2^{10}.3^4.5^2.7}$, [9.10]

$$(9.30)\quad 2\tau(n) \equiv 77n\sigma_9(n)-350(2n^2+n)\sigma_7(n) +210(9n^3+23n^2-14n)\sigma_5(n) -10(144n^4+1638n^3-1449n^2 +245n)\sigma_3(n)+175(72n^4-72n^3 +18n^2-n)\sigma(n) \pmod{2^{11}.3^4.5^2.7}, \quad [9.7]$$

$$(9.31)\quad 4\tau(n) \equiv -231n\sigma_9(n)+350(5n^2+12n)\sigma_7(n) -630(6n^3+64n^2-35n)\sigma_5(n) +60(36n^4+1848n^3-1617n^2 +280n)\sigma_3(n)-105(720n^4 -744n^3+192n^2-11n)\sigma(n) \pmod{2^{11}.3^5.5^2.7}, \quad [9.10]$$

$$(9.32)\quad 24\tau(n) \equiv -11n\sigma_9(n)+280n\sigma_5(n) -210n^2\sigma_3(n)-35n\sigma(n) \pmod{2^6.3^2.5^3.7}, \quad [9.2]$$

$$(9.33)\quad 36\tau(n) \equiv 11n\sigma_9(n)-735n\sigma_5(n) +20(63n^2-25n)\sigma_3(n) \pmod{2^4.3^3.5^3.7}, \quad [9.1]$$

$$(9.34)\quad 24\tau(n) \equiv -11n\sigma_9(n)+250n\sigma_7(n) -30(20n^2-21)\sigma_5(n)-70(3n^2+25n)\sigma_3(n) +35(60n^2-11n)\sigma(n) \pmod{2^7.3^3.5^3.7}, \quad [9.2]$$

$$(9.35)\quad 36\tau(n) \equiv 11n\sigma_9(n)-125n\sigma_7(n) +15(10n^2-49n)\sigma_5(n)+5(252n^2 +425n)\sigma_3(n)-50(63n^2-10n)\sigma(n) \pmod{2^6.3^4.5^3.7}; \quad [9.1]$$

$$(11.1)\quad 1800\tau(n) \equiv -273\sigma_{11}(n)+691[2\sigma_7(n)+\sigma_3(n)] \pmod{2^5.3.5.691}, \quad [11.3]$$

$$(11.2)\quad 1800\tau(n) \equiv -273\sigma_{11}(n)+691[11\sigma_9(n) -2(15n-11)\sigma_7(n)+\sigma_3(n)-\sigma(n)] \pmod{2^6.3^2.5.691}, \quad [11.3]$$

$$(11.3)\quad 600\tau(n) \equiv -91\sigma_{11}(n)+691[2\sigma_7(n)-\sigma_3(n)] \pmod{2^9.3.5^2.691}, \quad [11.2]$$

$$(11.4)\quad 776\tau(n) \equiv -6825\sigma_{11}(n)+691[11(6n-5)\sigma_9(n) -50(2n^2-3n+1)\sigma_7(n)+25\sigma_3(n) -5(6n-1)\sigma(n)] \pmod{2^7.3^2.5^2.691}, \quad [11.5]$$

$$(11.5)\quad 1800\tau(n) \equiv -273\sigma_{11}(n)+691[6\sigma_7(n)-3\sigma_3(n)] \pmod{2^8.3^2.5^2.691}, \quad [11.3]$$

$$(11.6)\quad 1800\tau(n) \equiv -273\sigma_{11}(n)+691[11\sigma_9(n)-2(15n-13)\sigma_7(n) -42\sigma_5(n)+(60n-23)\sigma_3(n)+\sigma(n)] \pmod{2^9.3^3.5^2.691}, \quad [11.2, 11.3]$$

$$(11.7)\quad 1800\tau(n) \equiv -273\sigma_{11}(n)+691[22\sigma_9(n)-2(30n-23)\sigma_7(n) -84\sigma_5(n)+(120n-24043)\sigma_3(n) +2(14400n-2399)\sigma(n)] \pmod{2^{10}.3^4.5^2.691}, \quad [11.2]$$

$$\begin{aligned}(11.8)\qquad 600\tau(n) \equiv{}& -91\sigma_{11}(n)+691[11\sigma_9(n)-2(15n+89)\sigma_7(n)\\ &+14(60n+17)\sigma_5(n)\\ &-(720n^2+2340n-979)\sigma_3(n)\\ &+(2400n^2-1320n+121)\sigma(n)] \quad (\text{mod } 2^{11}.3^4.5^3.691), \quad [11.2]\end{aligned}$$

$$\begin{aligned}(11.9)\qquad 776\tau(n) \equiv{}& -6825\sigma_{11}(n)+691[11(6n-5)\sigma_9(n)\\ &-50(2n^2-3n-1)\sigma_7(n)\\ &-210(2n-1)\sigma_5(n)+5(72n^2-60n+5)\sigma_3(n)\\ &+5(6n-1)\sigma(n)] \quad (\text{mod } 2^{10}.3^3.5^3.691), \quad [11.5, 11.7]\end{aligned}$$

$$\begin{aligned}(11.10)\qquad 776\tau(n) \equiv{}& -6825\sigma_{11}(n)+691[22(3n+10)\sigma_9(n)\\ &-50(2n^2+42n-31)\sigma_7(n)\\ &+60(90n^2-112n+21)\sigma_5(n)\\ &-5(720n^3-1152n^2+60n+95)\sigma_3(n)\\ &-10(180n^2-93n+8)\sigma(n)] \quad (\text{mod } 2^{11}.3^4.5^3.691), \quad [11.7)\end{aligned}$$

$$\begin{aligned}(11.11)\qquad 776\tau(n) \equiv{}& -6825\sigma_{11}(n)+691[11(6n+45)\sigma_9(n)\\ &-50(2n^2+87n+39)\sigma_7(n)\\ &+30(360n^2+966n-623)\sigma_5(n)\\ &-15(480n^3+6456n^2-5980n+1065)\sigma_3(n)\\ &+15(4800n^3-5040n^2+1322n-77)\sigma(n)] \quad (\text{mod } 2^{13}.3^5.5^3.691), \quad [11.7]\end{aligned}$$

$$(11.12)\qquad 756\tau(n) \equiv 65\sigma_{11}(n)+691\sigma_5(n) \quad (\text{mod } 2^3.3^3.7.691), \quad [11.1]$$

$$\begin{aligned}(11.13)\qquad 1800\tau(n) \equiv{}& -273\sigma_{11}(n)+691[2\sigma_7(n)+42\sigma_5(n)\\ &-3(20n-7)\sigma_3(n)-2\sigma(n)] \quad (\text{mod } 2^6.3^3.5.7.691), \quad [11.3]\end{aligned}$$

$$\begin{aligned}(11.14)\qquad 1512\tau(n) \equiv{}& 130\sigma_{11}(n)-691[20\sigma_7(n)-(42n-19)\sigma_5(n)\\ &+\sigma(n)] \quad (\text{mod } 2^4.3^3.5.7.691), \quad [11.1]\end{aligned}$$

$$\begin{aligned}(11.15)\qquad 15120\tau(n) \equiv{}& 1300\sigma_{11}(n)-691[11\sigma_9(n)-41\sigma_5(n)\\ &+10\sigma_3(n)] \quad (\text{mod } 2^7.3^3.5.7.691), \quad [11.1]\end{aligned}$$

$$\begin{aligned}(11.16)\qquad 1800\tau(n) \equiv{}& -273\sigma_{11}(n)+691[11\sigma_9(n)-2(15n-11)\sigma_7(n)\\ &+42\sigma_5(n)-(60n+179)\sigma_3(n)\\ &+(240n-43)\sigma(n)] \quad (\text{mod } 2^7.3^3.5.7.691), \quad [11.8]\end{aligned}$$

$$\begin{aligned}(11.17)\qquad 37800\tau(n) \equiv{}& -5733\sigma_{11}(n)+691[22\sigma_9(n)\\ &+42\sigma_7(n)-42\sigma_5(n)+41\sigma_3(n)] \quad (\text{mod } 2^8.3^3.5.7.691), \quad [11.3]\end{aligned}$$

$$\begin{aligned}(11.18)\qquad 404\tau(n) \equiv{}& 4550\sigma_{11}(n)-691[11(6n-5)\sigma_9(n)\\ &-100(2n^2-3n+1)\sigma_7(n)\\ &+10(18n^3-36n^2+21n-7)\sigma_5(n)\\ &+50(3n-1)\sigma_3(n)-5(24n^2\\ &-12n+1)\sigma(n)] \quad (\text{mod } 2^9,3^3,5.7,691), \quad [11.8]\end{aligned}$$

$$
\begin{aligned}
(11.19)\qquad 7560\tau(n) \equiv{}& 650\sigma_{11}(n)-691[11\sigma_9(n)\\
&-21\sigma_7(n)-31\sigma_5(n)+31\sigma_3(n)] \pmod{2^9.3^4.5.7.691}, \quad [11.1]
\end{aligned}
$$

$$
\begin{aligned}
(11.20)\qquad 37800\tau(n) \equiv{}& -5733\sigma_{11}(n)+691[253\sigma_9(n)\\
&-2(315n-31])\sigma_7(n)\\
&+42(20n-11)\sigma_5(n)\\
&+41\sigma_3(n)-41\sigma(n)] \pmod{2^{11}.3^4.5.7.691}, \quad [11.3]
\end{aligned}
$$

$$
\begin{aligned}
(11.21)\qquad 808\tau(n) \equiv{}& 9100\sigma_{11}(n)-691[33(4n-1)\sigma_9(n)\\
&-10(40n^2+3n-22)\sigma_7(n)\\
&+2(180n^3+396n^2-672n+77)\sigma_5(n)\\
&-12(84n^3-126n^2-25n+20)\sigma_3(n)\\
&-31(24n^2-12n+1)\sigma(n)] \pmod{2^{13}.3^4.5.7.691}, \quad [11.8]
\end{aligned}
$$

$$
\begin{aligned}
(11.22)\qquad 32\tau(n) \equiv{}& 15925\sigma_{11}(n)-691[77(6n-5)\sigma_9(n)\\
&-1750(2n^2-3n+1)\sigma_7(n)\\
&+350(36n^3-72n^2+42n-7)\sigma_5(n)\\
&-25(864n^4-2016n^3+1512n^2\\
&-420n+35)\sigma_3(n)+7(1728n^5\\
&-4320n^4+3608n^3-1200n^2\\
&+150n-5)\sigma(n)] \pmod{2^{14}.3^4.5.7.691}, \quad [11.10]
\end{aligned}
$$

$$
\begin{aligned}
(11.23)\qquad 9504\tau(n) \equiv{}& 13650\sigma_{11}(n)-691[11(6n-5)\sigma_9(n)\\
&+100\sigma_7(n)-210n\sigma_5(n)\\
&+50(3n-1)\sigma_3(n)+5\sigma(n)] \pmod{2^7.3^3.5^2.7.691}, \quad [11.6]
\end{aligned}
$$

$$
\begin{aligned}
(11.24)\qquad 776\tau(n) \equiv{}& -6825\sigma_{11}(n)+691[11(6n\\
&-5)\sigma_0(n)-50(2n^2-3n+1)\sigma_7(n)\\
&+1050\sigma_5(n)-25(180n-61)\sigma_3(n)\\
&+5(720n^2-366n+31)\sigma(n)] \pmod{2^6.3^3.5^2.7.691}, \quad [11.5]
\end{aligned}
$$

$$
\begin{aligned}
(11.25)\qquad 756\tau(n) \equiv{}& 65\sigma_{11}(n)-691[20\sigma_7(n)-2(21n-10)\sigma_5(n)\\
&-105\sigma_3(n)+2(63n-10)\sigma(n)] \pmod{2^4.3^4.5^2.7.691}, \quad [11.1]
\end{aligned}
$$

$$
\begin{aligned}
(11.26)\qquad 15120\tau(n) \equiv{}& 1300\sigma_{11}(n)-691[11\sigma_9(n)+200\sigma_7(n)\\
&-4(105n+68)\sigma_5(n)+10(63n-20)\sigma_3(n)\\
&+31\sigma(n)] \pmod{2^8.3^4.5^2.7.691}, \quad [11.1]
\end{aligned}
$$

$$
\begin{aligned}
(11.27)\qquad 9504\tau(n) \equiv{}& 13650\sigma_{11}(n)-691[33(2n+15)\sigma_9(n)\\
&-300(10n-7)\sigma_7(n)+30(120n^2-147n\\
&+35)\sigma_5(n)+150(n+3)\sigma_3(n)\\
&-15(40n-7)\sigma(n)] \pmod{2^8.3^4.5^2.7.691}, \quad [11.6]
\end{aligned}
$$

$$\begin{aligned}(11.28)\quad 16296\tau(n) \equiv & -143325\sigma_{11}(n)+691[11(126n-55)\sigma_9(n)\\ & -50(42n^2-8n-19)\sigma_7(n)\\ & +150(24n^3-28n+7)\sigma_5(n)\\ & +1025\sigma_3(n)-205(6n-1)\sigma(n)] \pmod{2^{10}.3^4.5^2.7.691}, \quad [11.5\end{aligned}$$

$$\begin{aligned}(11.29)\quad 404\tau(n) \equiv & \ 4550\sigma_{11}(n)-691[11(6n-5)\sigma_9(n)\\ & -100(2n^2-3n+8)\sigma_7(n)\\ & +10(18n^3-36n^2-609n-301)\sigma_5(n)\\ & -10(1512n^2-1275n+215)\sigma_3(n)\\ & +5(2016n^3-2040n^2+516n-29)\sigma(n)] \pmod{2^{10}.3^4.5^2.7.691}, \quad [11.8\end{aligned}$$

$$\begin{aligned}(11.30)\quad 19008\tau(n) \equiv & \ 27300\sigma_{11}(n)-691[11(12n+11)\sigma_9(n)\\ & -10(63n-62)\sigma_7(n)-42(10n+21)\sigma_5(n)\\ & +520(3n-1)\sigma_3(n)+81\sigma(n)] \pmod{2^{11}.3^4.5^2.7.691}, \quad [11.6\end{aligned}$$

$$\begin{aligned}(11.31)\quad 96\tau(n) \equiv & \ 47775\sigma_{11}(n)-691[154(6n-5)\sigma_9(n)\\ & -350(12n^2-18n+5)\sigma_7(n)\\ & +840(9n^3-18n^2+7n)\sigma_5(n)\\ & -5(864n^4-2016n^3+840n-175)\sigma_3(n)\\ & -70(72n^3-72n^2+18n-1)\sigma(n)] \pmod{2^{12}.3^4.5^2.7.691}, \quad [11.9]\end{aligned}$$

$$\begin{aligned}(11.32)\quad 19008\tau(n) \equiv & \ 27300\sigma_{11}(n)-691[33(4n+37)\sigma_9(n)\\ & -30(221n-14)\sigma_7(n)+6(1200n^2\\ & +1470n-1267)\sigma_5(n)-240(63n^2\\ & -59n-2)\sigma_3(n)-93(40n-7)\sigma(n)] \pmod{2^{12}.3^5.5^2.7.691}, \quad [11.6]\end{aligned}$$

$$\begin{aligned}(11.33)\quad 96\tau(n) \equiv & \ 47775\sigma_{11}(n)-691[231(4n+5)\sigma_9(n)\\ & -1050(4n^2+19n-15)\sigma_7(n)\\ & +210(36n^3+528n^2-672n+175)\sigma_5(n)\\ & -15(288n^4+16128n^3-25200n^2\\ & +10780n-1225)\sigma_3(n)+105(1440n^4\\ & -2448n^3+1248n^2-212n+9)\sigma(n)] \pmod{2^{13}.3^5.5^2.7.691}, \quad [11.9]\end{aligned}$$

$$\begin{aligned}(11.34)\quad 9504\tau(n) \equiv & \ 13650\sigma_{11}(n)-691[11(6n-5)\sigma_9(n)\\ & +2200\sigma_7(n)-210(43n-21)\sigma_5(n)\\ & +10(756n^2-615n-5)\sigma_3(n)\\ & +5(252n-41)\sigma(n)] \pmod{2^8.3^4.5^3.7.691}, \quad [11.6]\end{aligned}$$

$$\begin{aligned}(11.35)\quad 9504\tau(n) \equiv & \ 13650\sigma_{11}(n)-691[33(2n+15)\sigma_9(n)-600(5n\\ & -7)\sigma_7(n)+30(120n^2-441n-553)\sigma_5(n)\\ & +30(252n^2+2945n-1085)\sigma_3(n)\\ & -15(5040n^2-2564n+217)\sigma(n)] \pmod{2^9.3^5.5^3.7.691}, \quad [11.6]\end{aligned}$$

$$(11.36)\quad 9504\tau(n) \equiv 13650\sigma_{11}(n)-691[11(6n-5)\sigma_9(n) \\ -5\sigma(n)] \pmod{2^3.3.5.11.691}, \quad [11.4]$$

$$(11.37)\quad 38016\tau(n) \equiv 54600\sigma_{11}(n)-691[33(8n-3)\sigma_9(n) \\ -110(3n-2)\sigma_7(n)-31\sigma(n)] \pmod{2^8.3^2.5.11.691}, \quad [11.4]$$

$$(11.38)\quad 9504\tau(n) \equiv 13650\sigma_{11}(n)-691[11(6n-5)\sigma_9(n) \\ +550\sigma_3(n)-15(44n-7)\sigma(n)] \pmod{2^4.3^2.5^2.11.691}, \quad [11.4]$$

$$(11.39)\quad 19008\tau(n) \equiv 27300\sigma_{11}(n)-691[22(6n-5)\sigma_9(n) \\ +231\sigma_5(n)-110(3n-1)\sigma_3(n)-21\sigma(n)] \pmod{2^7.3^3.5.7.11.691}, \quad [11.4]$$

$$(11.40)\quad 199584\tau(n) \equiv 286650\sigma_{11}(n)-691[231(6n-5)\sigma_9(n) \\ +1100\sigma_7(n)-1155(2n-1)\sigma_5(n) \\ -50\sigma(n)] \pmod{2^7.3^3.5^2.7.11.691}; \quad [11.4]$$

$$(13.1)\quad 30240\tau(n) \equiv -691\sigma_{13}(n)+2600\sigma_{11}(n) \\ -691[20\sigma_7(n)-61\sigma_5(n)] \pmod{2^8.3^3.5.7.691}, \quad [13.1]$$

$$(13.2)\quad 37800\tau(n) \equiv 691\sigma_{13}(n)-5733\sigma_{11}(n) \\ +691[62\sigma_7(n)-21\sigma_5(n)+21\sigma_3(n)] \pmod{2^8.3^3.5.7.691}, \quad [13.1]$$

$$(13.3)\quad 7560\tau(n) \equiv 691\sigma_{13}(n)-8333\sigma_{11}(n) \\ +691[22\sigma_9(n)+62\sigma_7(n) \\ -103\sigma_5(n)+41\sigma_3(n)] \pmod{2^{11}.3^4.5.7.691}, \quad [13.1]$$

$$(13.4)\quad 30240\tau(n) \equiv -691\sigma_{13}(n)+2600\sigma_{11}(n) \\ +691[231\sigma_9(n)-630n\sigma_7(n) \\ +(840n-359)\sigma_5(n)-41\sigma(n)] \pmod{2^9.3^4.5^2.7.691}, \quad [13.1]$$

$$(13.5)\quad 19008\tau(n) \equiv -691\sigma_{13}(n)+27300\sigma_{11}(n) \\ -691[132(n-1)\sigma_9(n) \\ +220\sigma_7(n)-21(20n-1)\sigma_5(n) \\ +60(5n-2)\sigma_3(n)+10\sigma(n)] \pmod{2^9.3^4.5^2.7.691}, \quad [13.3]$$

$$(13.6)\quad 37800\tau(n) \equiv 691\sigma_{13}(n)-5733\sigma_{11}(n)-691[22\sigma_9(n) \\ -146\sigma_7(n)-21\sigma_5(n)+83\sigma_3(n)] \pmod{2^{11}.3^4.5^2.7.691}, \\ [13.1,\ 13.3]$$

$$(13.7)\quad 38016\tau(n) \equiv -691\sigma_{13}(n)+54600\sigma_{11}(n) \\ -691[66(4n+13)\sigma_9(n)-20(300n \\ -221)\sigma_7(n)+3(2400n^2+3080n \\ +707)\sigma_5(n)+60(10n+13)\sigma_3(n) \\ -20(60n-11)\sigma(n)] \pmod{2^{10},3^5,5^2.7.691}, \quad [13.3]$$

$$\begin{aligned}(13.8)\quad 18792\tau(n) \equiv {} & 691\sigma_{13}(n)-33033\sigma_{11}(n)\\ & +691[33(4n+3)\sigma_9(n)-2(315n\\ & -383)\sigma_7(n)-21(20n+41)\sigma_5(n)\\ & +3(520n-201)\sigma_3(n)+31\sigma(n)] \pmod{2^{12}.3^5.5^2.7.691}, \quad [13.3]\end{aligned}$$

$$\begin{aligned}(13.9)\quad 216\tau(n) \equiv {} & -691\sigma_{13}(n)+60333\sigma_{11}(n)\\ & -691[264(n+5)\sigma_9(n)-2(3630n\\ & -593)\sigma_7(n)+3(2400n^2+2800n\\ & -2821)\sigma_5(n)-3(5040n^2-5240n\\ & +41)\sigma_3(n)-2(1860n-341)\sigma(n)] \pmod{2^{13}.3^6.5^2.7.691}, \quad [13.3]\end{aligned}$$

$$\begin{aligned}(13.10)\quad 19008\tau(n) \equiv {} & -691\sigma_{13}(n)+27300\sigma_{11}(n)\\ & -691[33(4n-11)\sigma_9(n)+10(63n\\ & +400)\sigma_7(n)-21(860n-463)\sigma_5(n)\\ & +60(252n^2-226n+5)\sigma_3(n)\\ & +(2520n-431)\sigma(n)] \pmod{2^{10}.3^5.5^3.7.691}, \quad [13.3]\end{aligned}$$

$$\begin{aligned}(13.11)\quad 38016\tau(n) \equiv {} & -691\sigma_{13}(n)+54600\sigma_{11}(n)\\ & -691[33(8n+19)\sigma_9(n)\\ & -30(179n-200)\sigma_7(n)+3(2400n^2\\ & -5880n-12299)\sigma_5(n)+60(120n^2\\ & +3039n-1030)\sigma_3(n)-(151200n^2\\ & -75600n+6301)\sigma(n)] \pmod{2^{11}.3^6.5^3.7.691}, \quad [13.3]\end{aligned}$$

$$\begin{aligned}(13.12)\quad 37800\tau(n) \equiv {} & 691\sigma_{13}(n)-5733\sigma_{11}(n)\\ & -691[338\sigma_7(n)-21(40n\\ & +21)\sigma_5(n)+63(20n-7)\sigma_3(n)\\ & +62\sigma(n)] \pmod{2^9.3^4.5.7^2.691}, \quad [13.1]\end{aligned}$$

$$\begin{aligned}(13.13)\quad 19008\tau(n) \equiv {} & -691\sigma_{13}(n)+27300\sigma_{11}(n)\\ & -691[11(12n-11)\sigma_9(n)\\ & +10\sigma_3(n)-10\sigma(n)] \pmod{2^5.3^2.5.11.691}, \quad [13.2]\end{aligned}$$

$$\begin{aligned}(13.14)\quad 19800\tau(n) \equiv {} & 691.2\sigma_{13}(n)-3003\sigma_{11}(n)\\ & -691[22\sigma_9(n)-22\sigma_7(n)\\ & -31\sigma(n)] \pmod{2^4.3^2.5.11.691}, \quad [13.2]\end{aligned}$$

$$\begin{aligned}(13.15)\quad 18216\tau(n) \equiv {} & -691.2\sigma_{13}(n)+57603\sigma_{11}(n)\\ & -691[11(24n-11)\sigma_9(n)\\ & -22(15n-11)\sigma_7(n)\\ & +31\sigma_3(n)-31\sigma(n)] \pmod{2^9.3^3.5.11.691}, \quad [13.2]\end{aligned}$$

$$(13.16)\quad 19008\tau(n) \equiv -691\sigma_{13}(n)+27300\sigma_{11}(n) -691[11(12n-11)\sigma_9(n) -231\sigma_5(n)+10(33n +100)\sigma_3(n)-(1320n -221)\sigma(n)] \pmod{2^6.3^3.5^3.11.691}, \quad [13.2]$$

$$(13.17)\quad 19008\tau(n) \equiv -691\sigma_{13}(n)+27300\sigma_{11}(n) -691[11(12n-11)\sigma_9(n) -22\sigma_7(n)+231\sigma_5(n) -2(165n-71)\sigma_3(n) -21\sigma(n)] \pmod{2^5.3^4.5.7.11.691}, \quad [13.2]$$

$$(13.18)\quad 399168\tau(n) \equiv -691.21\sigma_{13}(n)+578300\sigma_{11}(n) -691[22(126n-121)\sigma_9(n) +2200\sigma_7(n)-231(20n-11)\sigma_5(n) +100\sigma_3(n)-100\sigma(n)] \pmod{2^9.3^4.5^2.7\ 11.691}, \quad [13.2]$$

$$(13.19)\quad 5189184\tau(n) \equiv -11.691^2\sigma_{13}(n) +2730(11n+2719)\sigma_{11}(n) -691[6006(6n-5)\sigma_9(n) -10331\sigma(n)] \pmod{2^7.3^3.5.7.11.13.691}. \quad [13.4]$$

It is obvious from Tables C(1) and C(2) that it is quite simple to deduce relations of the type

$$A\sigma_k(n) \equiv 0,\quad A\sigma_k(n) \equiv B\sigma_l(n),$$
$$A\tau(n) \equiv 0,\quad A\tau(n) \equiv B\sigma_k(n),$$

where A's and B's are polynomial in n, but the corresponding moduli are in general weaker than those from which they are deduced.

4. Finally it follows from the well-known result, $\sigma_k(n) \equiv 0$, k odd, for almost all n with respect to any modulus g, and the congruences (7.44), (9.8), (11.1) of Table C(2) that

$$(30)\qquad \tau(n) \equiv 0 \pmod{2^{10}.3^5.5^3.7.691},$$

for almost all n.

I should like to put on record my deep sense of obligation to Mr R. C. Bose for his very kind and constant encouragement during the preparation of this paper.

STATISTICAL LABORATORY,
PRESIDENCY COLLEGE,
CALCUTTA

Reference

Lahiri, D.B., (1946), *Bull. Cal. Math. Soc.*, **38**, 193-206.

CALCUTTA MATHEMATICAL SOCIETY

Report of the Council for the year 1946 to the Annual General Meeting of the Calcutta Mathematical Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1946, as required by the provisions of Rule 25.

The Council :—The Council of the Society for the year 1946 consisting of the officers and members elected at the last Annual General Meeting together with the Editorial Secretary, was constituted as follows :—

President

Prof. N. M. Basu

Vice-Presidents

Prof. A. C. Banerjee
Dr. R. N. Sen
Prof. N. R. Sen
Dr. S. S. Pillai
Prof. M. R. Siddiqi

Treasurer

Mr. S. C. Ghosh

Secretary

Mr. U. R. Burman

Editorial Secretary

Mr. S. Gupta

Other Members of the Counci

Prof. V. V. Narlikar
Prof. C. V. H. Rao
Prof. F. W. Levi
Dr. N. G. Shabde
Mr. N. N Ghosh
Prof. B. B. Sen
Prof. S. N. Bose
Mr. A. C. Choudhury
Dr. S. Ghosh
Mr. B. C. Chatterjee
Dr. Gorakh Prasad
Mr. P. K. Ghosh (*Assistant Secretary*)

General :—The various activities of the Society which have for the past few years been carried on in accordance with the usual wartime arrangements have had to be conducted on the same lines throughout this year also.

Membership :—The Council records with regret the death of one of the distinguished members of the society during the year 1946.

Prof. P. N. Ghosh, Head of the Department of Applied Physics, University College of Science & Technology, Calcutta.

During the year under review 8 new members were elected.

Meetings durings 1946 :—The Council held three meetings during the year and there were five ordinary general meetings which were devoted to the reading of papers communicated to the Society.

Publications :—Four numbers of the Bulletin were published during the year 1946, *viz.*, No. 4 of Vol. 37, and Nos. 1. 2 and 3 of Vol. 38. The tragic happenings that had overtaken the city of Calcutta in the middle of August, 1946, adversely affected the various activities of the Society and particularly the publication ; and inspite of best efforts, it was not possible to bring out more than three issues of Vol. 38 of the Bulletin, during the year under review.

The Society received with grateful thanks, in 1946, a grant of Rs. 400/- from the Rockefeller Foundation Grant for the improvement of Scientific Journals in India. The Council reports with pleasure that this amount has been utilised partly in the purchase of paper and partly in procuring certain essential mathematical types, which have largely contributed towards the improvement of the Bulletin.

The authorities of the Calcutta University have continued to lend their usual magnanimous services in printing the Bulletin free of charge, and the officers and members of the staff of the University Press have given their every sympathetic and active co-operation in bringing out the Bulletin in these difficult times. The council takes this opportunity of offering them its very sincere thanks.

Exchange of Publications :—The distribution of Bulletins to countries with which communication was impracticable during the war period has been carried on during the year 1946 with increasing facilities. Still it will not be immediately possible to make up the war-time gaps for all the Institutions on the exchange list though a few complete sets of wartime publications are being sent out as channels become available. The Council reports with pleasure that the Society has established exchange relations with eleven new Institutions during the year.

The Library :—The use of the Library continues to increase and during the year under review 200 books and journals were borrowed as compared with the figure of 170 in the previous year.

Finance :—The Annual Accounts of the Society for the year 1946 have been presented to the Council in the usual form by the auditors, Dr. B. S. Ray and Mr. N. L. Ghosh, who deserve the Council's grateful thanks for their honorary services. The council has the pleasure to report that in view of the Government decision to discontinue payment of interest at the rate of 3½ p. c. on the G. P. Notes and in consideration of the various schemes offered by the Government for dealing with these Notes, the Council decided to convert the G. P. Notes in both the General and K. K. G. P. Funds of the Society into 3 p. c. Notes, redeemable in 1986. The Council desires to draw the attention of the members to one or two points in the Auditors' report. On the receipt side, the amount under the head sale proceeds includes the amounts realised from foreign subscription agencies, and on the disbursement side the expenditure under the head, books journals, covers the subscriptions for foreign journals for the years 1946-47 and also includes the cost of journals ordered for in 1944 and received this year. It is a matter of gratification to note that the foreign accounts of the Society, which could not be settled up during the war period, have all been set right during the year under review. The financial position of the Society as a whole remains much the same as in the past few years.

CALCUTTA MATHEMATICAL SOCIETY

RECEIPTS AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST DECEMBER, 1946

Receipts

			Rs. as. p.
1. Opening balance			
(a) Cash with Secretary ...		7 1 3	
(b) Balance at Bank			
(i) Imperial Bank of India	1292 2 6		
(ii) Do (K. K. G. P. Fund)	216 5 3		
(iii) Bengal Central Bank	258 10 0		
(iv) P. O. Savings Bank	569 12 3		
		2336 14 0	
(c) G. P. Notes (General Fund) (Face Value Rs. 6,000) ...		5663 11 6	
(d) G. P. Notes (K. K. G. P. Fund) (Face Value Rs. 2,000) ...		1937 7 9	
			9945 2 6
2. Membership Subscriptions ...		856 14 0	
3. Admission Fees		80 0 0	
4. Sale Proceeds ...		566 13 9	
5. Rockefeller Foundation Grant ...		400 0 0	
			1903 11 9
6. Interest			
(a) G. P. Notes (General Fund) ...		105 0 0	
(b) Do. (K. K. G. P. Fund) ...		70 0 0	
(c) P. O. Savings Bank ...		8 8 9	
			183 8 9
		Total Rs.	12,032 7 0

Disbursements

			Rs. as. p.
1. Establishment ...		592 0 0	
2. Meetings ...		73 9 6	
3. Books and Journals (including binding charges) ...		546 14 6	
4. Bulletins			
(a) Papers, Blocks and Types	544 3 0		
(b) Postage	98 4 3		
		642 7 3	
5. Printing and Stationery		71 1 6	
6. Postage		65 13 0	
7. Bank Charges		13 6 0	
8. Miscellaneous (including conveyance charges) ...		41 2 6	
			2046 6 3
Closing Balance			
(a) Cash with Secretary ...		2 1 9	
(b) Cash with Secretary (to be deposited with Bank) ...		20 0 0	
(c) Balance at Bank			
(i) Imperial Bank of India	1194 10 6		
(ii) Do (K. K. G. P. Fund)	283 13 3		
(iii) Bengal Central Bank	305 15 0		
(iv) P. O. Savings Bank	578 5 0		
		2362 11 9	
(d) G. P. Notes (General Fund) (Face Value Rs. 6,000) ...		5663 11 6	
(e) G. P. Notes (K. K. G. P. Fund) (Face Value Rs. 2,000) ...		1937 7 9	
			9986 0 9
		Total Rs.	12,032 7 0

To

The Members of the Calcutta Mathematical Society,

We have examined the above Balance Sheet with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

B. S. RAY
N. L. GHOSH
Auditors

CENTRAL LIBRARY UNIVERSITY

ON THE DIVISORS OF NUMBERS

By

D. P. Banerjee

(*Communicated by the Secretary—Received April 1, 1947*)

The excess of the number of divisors of a number n which have the form $4m+1$ over the number of divisors which have the form $4m+3$ is termed by Glaisher as $E(n)$. I shall consider here some new formulæ of this and other functions of divisors.

Following Lambert (MacMahon, 1921) we can prove that

$$\sum_1^\infty \frac{a^m q^m}{1-q^m} = \sum_1^\infty \left(\sum a^d\right) q^n,$$

where d is a divisor of n and a is any arbitrary quantity. Hence

$$\sum_1^\infty \frac{m a^m q^{m-1}}{1-q^m} = \sum_1^\infty \left(\sum d a^d\right) q^{n-1},$$

or

$$\sum_1^\infty a^m \log(1-q^m) = \sum_1^\infty \frac{\left(\sum d a^d\right)}{n} q^n.$$

Let $a = e^{ix}$, then

$$\log \prod_{m=1}^\infty (1-q^m)^{\cos mx} = \sum_1^\infty \frac{1}{n}\left(\sum d \cos dx\right) q^n, \tag{1}$$

$$\log \prod_{m=1}^\infty (1-q^m)^{\sin mx} = \sum_1^\infty \frac{1}{n}\left(\sum d \sin dx\right) q^n, \tag{2}$$

Let $x = \pi/2$, then

$$\log \frac{(1-q^4)(1-q^8)\ldots}{(1-q^2)(1-q^6)\ldots} = \sum_1^\infty \frac{1}{n}\left(\sum d \cos d\pi/2\right) q^n. \tag{3}$$

Now according to Glaisher $E(n) = \sum \sin d\,\pi/2$. Hence

$$\sum_1^\infty E(n) q^n = \sum_1^\infty \frac{q^m \sin m(\pi/2)}{1-q^m}.$$

Hence

$$\sum_1^\infty \frac{1}{n} E(n) q^n = \log \frac{(1-q)(1-q^5)\ldots}{(1-q^3)(1-q^7)\ldots}. \tag{4}$$

Let δ be the divisors of n which possess uneven conjugates. Then

$$\sum_1^\infty \frac{a^m q^m}{1-q^{2m}} = \sum_1^\infty \left(\sum \delta a^\delta\right) q^n. \tag{5}$$

Hence

$$2\sum_{1}^{\infty}\frac{1}{n}(\sum\delta a^{\delta})q^{n} = \log\prod_{m=1}^{\infty}\left(\frac{1+q^{m}}{1-q^{m}}\right)^{a^{m}}; \tag{6}$$

putting $a = e^{ix}$, we have

$$\log\prod_{1}^{\infty}\left(\frac{1+q^{m}}{1-q^{m}}\right)^{\cos mx} = 2\sum_{1}^{\infty}\frac{1}{n}(\sum\delta\cos\delta x)q^{n}, \tag{7}$$

$$\log\prod_{1}^{\infty}\left(\frac{1+q^{m}}{1-q^{m}}\right)^{\sin mx} = 2\sum_{1}^{\infty}\frac{1}{n}(\sum\delta\sin\delta x)q^{n}; \tag{8}$$

when $x = \pi/2$.

$$\log\frac{(1-q^{2})(1+q^{4})(1-q^{6})\dots}{(1+q^{2})(1-q^{4})(1+q^{6})\dots} = 2\sum_{1}^{\infty}\frac{1}{n}\left(\sum\delta\cos\delta\frac{\pi}{2}\right)q^{n}, \tag{9}$$

$$\log\frac{(1+q)(1-q^{3})(1+q^{5})\dots}{(1-q)(1+q^{3})(1-q^{5})\dots} = 2\sum_{1}^{\infty}\frac{1}{n}\left(\sum\delta\sin\delta\frac{\pi}{2}\right)q^{n}. \tag{10}$$

A. M. COLLEGE,
MYMENSINGH, BENGAL.

References

Glaisher, *Proc. Lond. Math. Soc., Ser. 1*, 15,
MacMahon, P. A., (1921), *Proc. Lond. Math, Soc., Ser. 2*, 19, 305.

PROPOSITIONS ON POLAR RECIPROCALS

By

B. K. SENGUPTA

(Communicated by the Secretary—Received February 19, 1947)

In plane geometry, to a point (tangent) of a curve C, there corresponds a tangent (point) of the polar reciprocal of C with respect to a given conic. If P is a point of C and Q the point of contact of the corresponding tangent, then P and Q are corresponding points.

The following propositions give certain properties of polar reciprocals.

Proposition 1. *The points on the polar reciprocals of a given curve with respect to a given system of confocal conics corresponding to the same point on the given curve lie on a right line perpendicular to the tangent to the given curve at the given point.*

Proof :—Let the equation of the given curve be $f(x, y)=0$. and that of the system of confocal conics be

$$(b+\lambda)x^2+(a+\lambda)y^2-(a+\lambda)(b+\lambda) = 0,$$

where λ is a variable; let $P(x', y')$ be a point on the given curve; if $Q(x, y)$ be the corresponding point on the polar reciprocal, then

$$x = (a+\lambda)m, \qquad y = (b+\lambda)n,$$

where

$$m = \frac{\partial f}{\partial x'} \Big/ \left(x'\frac{\partial f}{\partial x'}+y'\frac{\partial f}{\partial y'}\right), \quad n = \frac{\partial f}{\partial y'} \Big/ \left(x'\frac{\partial f}{\partial x'}+y'\frac{\partial f}{\partial y'}\right)$$

On eliminating λ between the two relations, the equation of the locus of (x, y), the point on the polar reciprocal, is found to be

$$nx-my-nm(a-b) = 0$$

or

$$\left(x'\frac{\partial f}{\partial x'}+y'\frac{\partial f}{\partial y'}\right)\left(x\frac{\partial f}{\partial y'}-y\frac{\partial f}{\partial x'}\right)-\frac{\partial f}{\partial x'}\cdot\frac{\partial f}{\partial y'}(a-b) = 0,$$

and this line is perpendicular to the tangent to the given curve at P, *viz.*,

$$x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'}-\left(x'\frac{\partial f}{\partial x'}+y'\frac{\partial f}{\partial y'}\right) = 0.$$

Proposition 2. *The centres of curvature for points on polar reciprocals of a given curve with respect to a series of concentric circles, one lying on each and all corresponding to the same point on the given curve, lie on a right line passing through the centre of the circles.*

Proof:—Let O be the centre of the concentric circles and $P(x, y)$ a point on the given curve $f(x, y)=0$; let the radii of the circles, with respect to which polar reciprocals of the given curve are taken, be k_i, $i=1, 2, \ldots$. Let ON be the perpendicular on the tangent to the given curve at P. N is then a point on the first positive pedal with respect to O of $f(x, y)=0$; let P_i, $i=1, 2, \ldots$, be points on ON such that $ON.OP_i=k_i^2$, so that they lie on the polar reciprocals, all corresponding to P. Let C_i, $i=1, 2, \ldots$, be the centres of curvature respectively for the points P_i on the polar reciprocals. If ϕ is the angle which the tangent at P makes with OP, the tangent at N to the first positive pedal makes with ON the same angle $\pi-\phi$, the polar reciprocal being the inverse of the first positive pedal; hence the tangents are all parallel and consequently the normals at those points to the respective curves are also parallel; let ON be r and $OP_i=r_i$; let the radii of curvature at ρ_i be $P_i=P_iC_i$. Now these radii of curvature are respectively equal to

$$-k_i^2 q(x^2+y^2)^{3/2}/(px-y)^3$$

where $p=dy/dx$, $q=d^2y/dx^2$ and x, y, p, q refer to P and are the same throughout. Hence

$$\rho_1:\rho_2:\ldots\ldots = k_1^2:k_2^2:\ldots\ldots = rr_1:rr_2:\ldots\ldots$$

that is

$$P_1C_1:P_2C_2::\ldots\ldots = OP_1:OP_2:\ldots\ldots$$

whence $C_1, C_2 \ldots\ldots$ lie on a right line passing through O, the centre of the concentric circles.

SRIKHANDA, BURDWAN,
BENGAL, INDIA.

A NOTE ON STUDENT'S *t* FOR PAIRED SAMPLES

By

H. K. NANDI

(*Communicated by Mr. R. C. Bose—Received May 23, 1947*).

When two series of sample values are given such that a sample value in one series corresponds uniquely to a sample value in the other, the problem of testing for the equality of the means (the distribution of the sample values being assumed normal) has been solved by either (I) considering the differences between the paired values as independent samples with mean zero apply t-test to these differences, or (II) regarding the two series as independent apply t-test with increased degrees of freedom. Fisher (1941) suggests that "a more precise comparison is obtainable by this method (*i.e.*, Test II) only if the corresponding values of the two series are positively correlated and only if they are correlated to a sufficient extent to counterbalance the loss of precision due to basing our estimate of variance upon fewer degrees of freedom."

In this note, the two tests I and II have been studied from the point of view of control of the two kinds of error and their relative advantages have been appraised. It is found: (*i*) while test I is an exact test, test II is only an approximate one whose exact distribution involves the correlation coefficient ρ; (*ii*) for only positive values of ρ the first kind of error is controlled, error being less than the level of significance; and (*iii*) it is only when ρ is small—for samples of five ρ must be less than 0.1—that test II scores over test I in controlling the second kind of error.

Notations. Let (x_1, x_2), (z_{1l}, z_{2l}), $(l = 1, 2, \ldots, n)$ be $(n+1)$ pairs of random samples from a bi-variate normal population with variances (σ_1^2, σ_2^2), correlation ρ and $E(z_{1l}) = E(z_{2l}) = 0$ while $E(x_1) = m_1$, $E(x_2) = m_2$.

Put
$$n s_{ij} = \sum_l z_{il} z_{jl} \quad (i, j = 1, 2) \tag{1}$$

$$t = \frac{x_1 - x_2}{\sqrt{(s_{11} + s_{22} - 2s_{12})}}, \qquad w = \frac{x_1 - x_2}{\sqrt{(s_{11} + s_{22})}}. \tag{2}$$

Then test I and II consist in referring t and w to student's distribution with n and $2n$ *d.f.* respectively. While t is well known to follow Student's distribution, we have to obtain the distribution of w.

Distribution of w. Let us assume $\sigma_1^2 = \sigma_2^2 = \sigma^2$ because we are primarily interested in investigating the effect of correlation on the tests.

Transform x_1, x_2, z_{1l}, z_{2l} to new variables X_1, X_2, Z_{1l}, Z_{2l} given by

$$\left.\begin{array}{ll} \sqrt{2}\, X_1 = x_1 - x_2 & \sqrt{2}\, X_2 = x_1 + x_2 \\ \sqrt{2}\, Z_{1l} = z_{1l} - z_{2l}, & \sqrt{2}\, Z_{2l} = z_{1l} + z_{2l} \end{array}\right\} \tag{3}$$

so that

$$\left.\begin{array}{l} V(X_1) = V(Z_{1l}) = \sigma^2(1-\rho) = \gamma_1 \\ V(X_2) = V(Z_{2l}) = \sigma^2(1+\rho) = \gamma_2 \end{array}\right\}. \tag{4}$$

Put
$$nS_1 = \sum_l Z_{1l}^2, \qquad nS_2 = \sum_l Z_{2l}^2 \tag{5}$$

Then
$$s_{11}+s_{22} = S_1+S_2, \qquad w = \frac{X_1\sqrt{2}}{\sqrt{(S_1+S_2)}}. \tag{6}$$

The variables X_1, X_2, Z_{1l}, Z_{2l} follow normal probability law independently, whose joint distribution is given by

$$\frac{1}{\sqrt{(2\pi\gamma_1)}}.\exp\{-X_1^2/2\gamma_1\}.\left(\frac{1}{2\pi\gamma_1}\right)^{n/2}.\exp\{-nS_1/2\gamma_1\}.\left(\frac{1}{2\pi\gamma_2}\right)^{n/2}.\exp\{-nS_2/2\gamma_2\}.\prod dX_1\prod dZ_{1l}dZ_{2l}. \tag{7}$$

Thus the joint distribution of S_1 and S_2 is

$$\left(\frac{n}{2\gamma_1}\right)^{n/2}.\frac{1}{\Gamma(n/2)}.\exp\{-nS_1/2\gamma_1\}.S_1^{(n-2)/2}dS_1.\left\{\frac{n}{2\gamma_2}\right\}^{n/2}.\frac{1}{\Gamma(n/2)}.\exp\{-nS_2/2\gamma_2\}.S_2^{(n-2)/2}dS_2. \tag{8}$$

wherefrom the distribution of $S=(S_1+S_2)/2$ is obtained in the following alternative forms:

$$df(S) = \left\{\frac{4n^2}{4\gamma_1\gamma_2}\right\}^{n/2}.\frac{1}{\Gamma(n)}.\exp\{-2nS/2\gamma_1\}S^{n-1}dS$$
$$\times\left\{1+\sum_{i=1}^{\infty}\frac{(n/2)(n/2+1)\ \ldots\ (n/2+i-1)}{n(n+1)\ \ldots\ (n+i-1)}.\frac{1}{i\,!}.\left\{\frac{2n\rho.2S}{2\sqrt{(\gamma_1\gamma_2)}}\right\}^i\right\} \tag{9}$$

$$= \left[\frac{1-\rho}{1+\rho}\right]^{n/2}\left[\left[\frac{2n}{2\gamma_1}\right]^n.\frac{1}{\Gamma(n)}.\exp\{-2nS/2\gamma_1\}S^{n-1}dS\right.$$
$$\left.+\sum_{i=1}^{\infty}\frac{n(n+2)\ \ldots\ (n+2i-2)}{i\,!}.\left\{\frac{\rho}{1+\rho}\right\}^i.\left\{\frac{2n}{2\gamma_1}\right\}^{n+i}.\frac{1}{\Gamma(n+i)}.\exp\{-2nS/2\gamma_1\}S^{n+i-1}dS\right]. \tag{10}$$

$$\therefore\quad \int_0^S df(S) = \left\{\frac{1-\rho}{1+\rho}\right\}^{n/2}\left[\int_0^S I_{2n}(S,\gamma_1)dS\right.$$
$$\left.+\sum_{i=1}\frac{n(n+2)\ \ldots\ (n+2i-2)}{i\,!}.\left\{\frac{\rho}{1+\rho}\right\}^i\int_0^{2nS/(2n+2i)} I_{2n+2i}(S,\gamma_1)dS\right] \tag{11}$$

where $I_n(S,\gamma)dS$ is the distribution of S, the estimate of variance γ of a normal population with n $d.f.$ Or,

$$\int_0^S df(S) = \left\{\frac{1+\rho}{1-\rho}\right\}^{n/2}\left[\int_0^S I_{2n}(S,\gamma_2)dS\right.$$
$$\left.+\sum_{i=1}^{\infty}\frac{n(n+2)\ \ldots\ (n+2i-2)}{i\,!}.\left\{\frac{-\rho}{1-\rho}\right\}^i\int_0^{2nS/(2n+2i)} I_{2n+2i}(S,\gamma_2)dS\right]. \tag{12}$$

If we multiply the expressions (11) and (12) by the distribution function of X_1, we get the distribution of w in the following alternative forms:

$$\int_0^w \phi(w)dw = \left\{\frac{1-\rho}{1+\rho}\right\}^{n/2}\left[\int_0^w T_{2n}(t)dt+\sum_{i=1}^{\infty}\frac{n(n+2)\ \ldots\ (n+2i-2)}{i\,!}.\left\{\frac{\rho}{1+\rho}\right\}^i.\int_0^{w\sqrt{(2n+2i)}/\sqrt{(2n)}} T_{2n+2i}(t)dt\right]. \tag{13}$$

$$\int_0^w \phi(w)dw = \left\{\frac{1+\rho}{1-\rho}\right\}^{n/2}\left[\int_0^u T_{2n}(t)dt + \sum_{i=1}^{\infty} \frac{n(n+2)\ldots(n+2i-2)}{i!}\cdot\left\{\frac{-\rho}{1-\rho}\right\}^i\cdot\int_0^{u\sqrt{(2n+2i)}/\sqrt{(2n)}} T_{2n+2i}(t)dt\right]. \tag{14}$$

where $T_{2n}(t)dt$ is student's distribution with $2n$ *d.f.* and $u = w\sqrt{(1+\rho)}/\sqrt{(1-\rho)}$. The form (13) is useful when ρ is positive while (14) will be used when ρ is negative.

Control of the first kind of error. Test I, of course, controls the first kind of error alright. But since the distribution of w given by (13) or (14) involves the nuisance parameter ρ and as such the significance levels of w obtained from those of t with $2n$ *d.f.* are inexact when ρ is different from zero, it remains to be seen how far test II controls this error.

Suppose t_α is the $\alpha\%$ level of significance obtained from t-table with $2n$ *d.f.* If the $\alpha\%$ level of significance of w is also taken to be t_α, the exact magnitude of the first kind of error is seen from (13) to be less than α when ρ is positive. When, however, ρ is negative, this error varies from α in both directions depending on n and ρ. The following table reveals the degree of approximation involved regarding α in referring w to the t-table Here we consider only one tail-end of the distribution and take small values of n, because for large values of n test I is superior.

TABLE I

The exact magnitude of the first kind
of error: $\alpha = 0.05$

ρ	$n=4$	$n=8$
-0.50	0.08	0.08
-0.14	0.06	0.06
$+0.14$	0.04	0.04
$+0.50$	0.02	0.01
$+0.80$	0.003	$\leqslant 0.003$

Power of the tests. Here it has not been possible to use the expressions corresponding to (13) or (14) because extensive tables for the power of the t-test are not available. So a sampling experiment had been undertaken to have some idea about the power for positive values of ρ only. 50 samples of five were drawn from a bivariate normal population with means zero and standard deviations unity. Mean difference of a magnitude which is detected by test I with probabilities 50%, 70% and 90% was added to one pair of observations and the value of w was calculated. Table II gives the percentage of cases in which test I detected the difference (the expected values of these percentages are, of course, those noted above), Table III gives the percentage of cases where w came out to have significant values when referred to t-table with 8 *d.f.*

TABLE II

Percentage of cases detected by test I
$n = 4$

ρ	Mean diff. in terms of power of test I.		
	50%	70%	90%
0.14	48	68	86
0.50	54	76	90
0.80	60	74	92

TABLE III

Percentage of cases detected
by test II

ρ	Mean diff. in terms of power of test I		
	50%	70%	90%
0.14	48	70	88
0.50	28	40	68
0.80	12	22	46

Table II reveals that observed values agreed well with the expected values and as such the observed values of the power of test II given in table III are not likely to be wide of the mark. Inspection of table III suggests that when $\rho=0.14$, the powers of both the tests are more or less the same and hence when $\rho < 0.1$, test II will score an advantage over test I from consideration of its power.

CALCUTTA STATISTICAL ASSOCIATION,
CALCUTTA

References

Fisher, R. A., 1941), *Statistical Methods for Research Workers*, eighth edition, 125.
Nandi, H. K., (1946), *Science and Culture*, 10, 240.
Neyman, J. and Tokarska, (1936), *J. A. S. A.*, 31, 322.

ON A GENERALISED CONTINUED FRACTION

By

S. D. Upadhyay and P. N. Das Gupta

(*Received May 14, 1947*)

A series of expressions for the numerator and denominator of any convergent of a continued fraction by means of matrices had been given by Milne-Thomson (1933 *a*, *b*). The authors while working on a problem related to reciprocal differences in connection with interpolation formulae had to consider continued fractions of a general type where the coefficients themselves represent continued fractions. The present paper is a result of studies made in that connection.

Notation

In the continued fraction

$$x \equiv a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n +} \cdots$$

where

$$a_1 = a_{11} + \frac{1}{a_{12} +} \frac{1}{a_{13} +} \cdots \frac{1}{a_{1n} +} \cdots \quad \cdots$$

$$a_2 = a_{21} + \frac{1}{a_{22} +} \frac{1}{a_{23} +} \cdots \frac{1}{a_{2n} +} \cdots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_n = a_{n1} + \frac{1}{a_{n2} +} \frac{1}{a_{n3} +} \cdots \frac{1}{a_{nn} +} \cdots$$

we denote the convergents

$$a_{11} + \frac{1}{a_{21}}, \quad a_{11} + \frac{1}{a_{21} +} \frac{1}{a_{31}}, \cdots$$

$$\left(a_{11} + \frac{1}{a_{12}}\right) + \frac{1}{a_{21}}, \quad a_{11} + \frac{1}{\left(a_{21} + \frac{1}{a_{22}}\right) +} \frac{1}{a_{31}}, \text{ etc.}$$

by

$$A\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}, \quad A\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\end{pmatrix}, \cdots$$

$$A\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \quad A\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 1\end{pmatrix}, \text{ etc.}$$

and the corresponding numerators by

$$p\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}, \quad p\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\end{pmatrix}, \cdots$$

$$p\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \quad p\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 1\end{pmatrix}, \text{ etc..}$$

We further denote the convergents

$$a_{11},\quad a_{11}+\frac{1}{a_{12}},\quad a_{11}+\frac{1}{a_{12}+}\,\frac{1}{a_{13}},\ \ldots$$

by

$$A\binom{1}{1},\quad A\binom{1}{2},\quad A\binom{1}{3},\ \cdots$$

and the corresponding numerators by

$$p\binom{1}{1},\quad p\binom{1}{2},\quad p\binom{1}{3},\ \cdots.$$

(A)

We have

$$p\begin{pmatrix}1&2\\1&2\end{pmatrix}=a_{22}p\begin{pmatrix}1&2\\1&1\end{pmatrix}+p\binom{1}{1}.$$

$$p\begin{pmatrix}1&2\\2&2\end{pmatrix}=a_{23}p\begin{pmatrix}1&2\\2&1\end{pmatrix}+p\binom{1}{2}.$$

$$p\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}=a_{31}p\begin{pmatrix}1&2\\1&1\end{pmatrix}+p\binom{1}{1}.$$

$$p\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}=a_{32}p\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}+p\begin{pmatrix}1&2\\1&1\end{pmatrix}.$$

In general

$$p\begin{pmatrix}1&2&3&\ldots&r\\k&l&m&\ldots&s\end{pmatrix}=a_{rs}p\begin{pmatrix}1&2&3&\ldots&r\\k&l&m&\ldots&(s-1)\end{pmatrix}+p\begin{pmatrix}1&2&3&\ldots&r\\k&l&m&\ldots&(s-2)\end{pmatrix}. \tag{1}$$

Since

$$p\begin{pmatrix}1&2\\1&2\end{pmatrix}=a_{22}\left[a_{21}p\binom{1}{1}+1\right]+p\binom{1}{1}.$$

$$\therefore\quad p\begin{pmatrix}1&2\\1&2\end{pmatrix}=\begin{vmatrix}a_{21}&1\\-1&a_{22}\end{vmatrix}p\binom{1}{1}+a_{22}. \tag{2}$$

Again,

$$p\begin{pmatrix}1&2\\1&3\end{pmatrix}=a_{23}p\begin{pmatrix}1&2\\1&2\end{pmatrix}+p\begin{pmatrix}1&2\\1&1\end{pmatrix}.$$

So,

$$p\begin{pmatrix}1&2\\1&3\end{pmatrix}=\begin{vmatrix}a_{21}&-1&0\\1&a_{22}&-1\\0&1&a_{23}\end{vmatrix}p\binom{1}{1}+\begin{vmatrix}a_{22}&-1\\1&a_{23}\end{vmatrix}. \tag{3}$$

Similarly we can shew

$$p\begin{pmatrix}1&2\\1&4\end{pmatrix}=\begin{vmatrix}a_{21}&-1&0&0\\1&a_{22}&-1&0\\0&1&a_{23}&-1\\0&0&1&a_{24}\end{vmatrix}p\binom{1}{1}+\begin{vmatrix}a_{22}&-1&0\\1&a_{23}&-1\\0&1&a_{24}\end{vmatrix}. \tag{4}$$

Also,

$$p\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}=\begin{vmatrix}a_{31}&-1\\1&a_{32}\end{vmatrix}p\begin{pmatrix}1&2\\1&1\end{pmatrix}+a_{32}p\binom{1}{1}$$

and

$$p\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}=a_{33}p\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}+p\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix} \tag{5}$$

So,

$$p\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}=\begin{vmatrix}a_{31}&-1&0\\1&a_{32}&-1\\0&1&a_{33}\end{vmatrix}p\begin{pmatrix}1&2\\1&1\end{pmatrix}+\begin{vmatrix}a_{32}&-1\\1&a_{33}\end{vmatrix}p\begin{pmatrix}1\\1\end{pmatrix}. \qquad (6)$$

Again since

$$p\begin{pmatrix}1&2&3\\1&1&4\end{pmatrix}=a_{34}p\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}+p\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}$$

we can similarly shew

$$p\begin{pmatrix}1&2&3\\1&1&4\end{pmatrix}=\begin{vmatrix}a_{31}&-1&0&0\\1&a_{32}&-1&0\\0&1&a_{33}&-1\\0&0&1&a_{34}\end{vmatrix}p\begin{pmatrix}1&2\\1&1\end{pmatrix}+\begin{vmatrix}a_{32}&-1&0\\1&a_{33}&-1\\0&1&a_{34}\end{vmatrix}p\begin{pmatrix}1\\1\end{pmatrix}. \qquad (7)$$

Also since

$$p\begin{pmatrix}1&2&3&4\\1&1&1&2\end{pmatrix}=a_{42}p\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}+p\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}$$

so,

$$p\begin{pmatrix}1&2&3&4\\1&1&1&2\end{pmatrix}=\begin{vmatrix}a_{41}&-1\\1&a_{42}\end{vmatrix}p\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}+a_{42}p\begin{pmatrix}1&2\\1&1\end{pmatrix}. \qquad (8)$$

And since

$$p\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}=a_{43}p\begin{pmatrix}1&2&3&4\\1&1&1&2\end{pmatrix}+p\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}$$

we can shew

$$p\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}=\begin{vmatrix}a_{41}&-1&0\\1&a_{12}&-1\\0&1&a_{43}\end{vmatrix}p\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}+\begin{vmatrix}a_{42}&-1\\1&a_{43}\end{vmatrix}p\begin{pmatrix}1&2\\1&1\end{pmatrix}. \qquad (9)$$

(B)

We have

$$p\begin{pmatrix}1&2\\1&5\end{pmatrix}=a_{25}p\begin{pmatrix}1&2\\1&4\end{pmatrix}+p\begin{pmatrix}1&2\\1&3\end{pmatrix}.$$

$$q\begin{pmatrix}1&2\\1&5\end{pmatrix}=a_{25}q\begin{pmatrix}1&2\\1&4\end{pmatrix}+q\begin{pmatrix}1&2\\1&3\end{pmatrix}.$$

$$\therefore\quad\begin{bmatrix}p\begin{pmatrix}1&2\\1&5\end{pmatrix}&p\begin{pmatrix}1&2\\1&4\end{pmatrix}\\q\begin{pmatrix}1&2\\1&5\end{pmatrix}&q\begin{pmatrix}1&2\\1&4\end{pmatrix}\end{bmatrix}=\begin{bmatrix}a_{25}p\begin{pmatrix}1&2\\1&4\end{pmatrix}+p\begin{pmatrix}1&2\\1&3\end{pmatrix}&p\begin{pmatrix}1&2\\1&4\end{pmatrix}\\a_{25}q\begin{pmatrix}1&2\\1&4\end{pmatrix}+q\begin{pmatrix}1&2\\1&3\end{pmatrix}&q\begin{pmatrix}1&2\\1&4\end{pmatrix}\end{bmatrix}.$$

$$=\begin{bmatrix}p\begin{pmatrix}1&2\\1&4\end{pmatrix}&p\begin{pmatrix}1&2\\1&3\end{pmatrix}\\q\begin{pmatrix}1&2\\1&4\end{pmatrix}&q\begin{pmatrix}1&2\\1&3\end{pmatrix}\end{bmatrix}\begin{bmatrix}a_{25}&1\\1&0\end{bmatrix}.$$

$$=\begin{bmatrix}p\begin{pmatrix}1&2\\1&3\end{pmatrix}&p\begin{pmatrix}1&2\\1&2\end{pmatrix}\\q\begin{pmatrix}1&2\\1&3\end{pmatrix}&q\begin{pmatrix}1&2\\1&2\end{pmatrix}\end{bmatrix}\begin{bmatrix}a_{24}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{23}&1\\1&0\end{bmatrix}.$$

$$= \begin{bmatrix} p\begin{pmatrix}1&2\\1&2\end{pmatrix} & p\begin{pmatrix}1&2\\1&1\end{pmatrix} \\ q\begin{pmatrix}1&2\\1&2\end{pmatrix} & q\begin{pmatrix}1&2\\1&1\end{pmatrix} \end{bmatrix} \begin{bmatrix} a_{23} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{24} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{35} & 1 \\ 1 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} p\begin{pmatrix}1&2\\1&1\end{pmatrix} & p\begin{pmatrix}1\\1\end{pmatrix} \\ q\begin{pmatrix}1&2\\1&1\end{pmatrix} & q\begin{pmatrix}1\\1\end{pmatrix} \end{bmatrix} \begin{bmatrix} a_{22} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{23} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{24} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{25} & 1 \\ 1 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} a_{11} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{21} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{22} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{23} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{24} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{25} & 1 \\ 1 & 0 \end{bmatrix}. \qquad (10)$$

Similarly,

$$\begin{bmatrix} p\begin{pmatrix}1&2\\1&5\end{pmatrix} \\ q\begin{pmatrix}1&2\\1&5\end{pmatrix} \end{bmatrix} = \begin{bmatrix} a_{11} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{21} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{22} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{23} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{24} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{25} \\ 1 \end{bmatrix}. \qquad (11)$$

More generally we have

$$\begin{bmatrix} p\begin{pmatrix}1&2\\1&n\end{pmatrix} & p\begin{pmatrix}1&2\\1&n-1\end{pmatrix} \\ q\begin{pmatrix}1&2\\1&n\end{pmatrix} & q\begin{pmatrix}1&2\\1&n-1\end{pmatrix} \end{bmatrix} = \begin{bmatrix} a_{11} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{21} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{22} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{2n} & 1 \\ 1 & 0 \end{bmatrix}. \qquad (12)$$

and

$$\begin{bmatrix} p\begin{pmatrix}1&2\\1&n\end{pmatrix} \\ q\begin{pmatrix}1&2\\1&n\end{pmatrix} \end{bmatrix} = \begin{bmatrix} a_{11} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{21} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{3n} \\ 1 \end{bmatrix}. \qquad (13)$$

We have

$$\begin{bmatrix} p\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} \\ q\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} \end{bmatrix} = \begin{bmatrix} p\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix} & p\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix} \\ q\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix} & q\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix} \end{bmatrix} \begin{bmatrix} a_{33} \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} p\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix} & p\begin{pmatrix}1&2\\1&2\end{pmatrix} \\ q\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix} & q\begin{pmatrix}1&2\\1&2\end{pmatrix} \end{bmatrix} \begin{bmatrix} a_{32} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{33} \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} p\begin{pmatrix}1&2\\1&2\end{pmatrix} & p\begin{pmatrix}1&2\\1&1\end{pmatrix} \\ q\begin{pmatrix}1&2\\1&2\end{pmatrix} & q\begin{pmatrix}1&2\\1&1\end{pmatrix} \end{bmatrix} \begin{bmatrix} a_{31} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{32} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{33} \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} a_{11} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{21} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{22} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{31} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{32} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{33} \\ 1 \end{bmatrix}. \qquad (14)$$

Similarly,

$$\begin{bmatrix} p\begin{pmatrix}1&2&3\\1&2&4\end{pmatrix} & p\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} \\ q\begin{pmatrix}1&2&3\\1&2&4\end{pmatrix} & q\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}a_{11}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{21}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{22}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{31}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{32}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{33}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{34}&1\\1&0\end{bmatrix}. \qquad (15)$$

More generally,

$$\begin{bmatrix} p\begin{pmatrix}1&2&3\\1&n&n\end{pmatrix} \\ q\begin{pmatrix}1&2&3\\1&n&n\end{pmatrix}\end{bmatrix}=\begin{bmatrix}a_{11}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{21}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{22}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{2n}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{31}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{32}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{3n}\\1\end{bmatrix}. \quad (16)$$

and

$$\begin{bmatrix} p\begin{pmatrix}1&2&3\\1&n&n\end{pmatrix} & p\begin{pmatrix}1&2&3\\1&n&n-1\end{pmatrix} \\ q\begin{pmatrix}1&2&3\\1&n&n\end{pmatrix} & q\begin{pmatrix}1&2&3\\1&n&n-1\end{pmatrix}\end{bmatrix}=\begin{bmatrix}a_{11}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{21}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{22}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{2n}&1\\1&0\end{bmatrix}$$

$$\times\begin{bmatrix}a_{31}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{32}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{33}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{3n}&1\\1&0\end{bmatrix}. \qquad (17)$$

We have,

$$\begin{bmatrix} p\begin{pmatrix}1&2&3\\n&n&n\end{pmatrix} \\ q\begin{pmatrix}1&2&3\\n&n&n\end{pmatrix}\end{bmatrix}=\begin{bmatrix}a_{11}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{12}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{13}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{1n}&1\\1&0\end{bmatrix}$$

$$\times\begin{bmatrix}a_{21}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{22}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{23}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{2n}&1\\1&0\end{bmatrix}$$

$$\times\begin{bmatrix}a_{31}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{32}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{33}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{3n}\\1\end{bmatrix}. \qquad (18)$$

and

$$\begin{bmatrix} p\begin{pmatrix}1&2&3\\n&n&n\end{pmatrix} & p\begin{pmatrix}1&2&3\\n&n&n-1\end{pmatrix} \\ q\begin{pmatrix}1&2&3\\n&n&n\end{pmatrix} & q\begin{pmatrix}1&2&3\\n&n&n-1\end{pmatrix}\end{bmatrix}=\begin{bmatrix}a_{11}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{12}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{13}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{1n}&1\\1&0\end{bmatrix}$$

$$\times\begin{bmatrix}a_{21}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{22}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{23}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{2n}&1\\1&0\end{bmatrix}$$

$$\times\begin{bmatrix}a_{31}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{32}&1\\1&0\end{bmatrix}\begin{bmatrix}a_{33}&1\\1&0\end{bmatrix}\cdots\begin{bmatrix}a_{3n}&1\\1&0\end{bmatrix}. \qquad (19)$$

More generally,

$$\begin{bmatrix} p\begin{pmatrix}1 & 2 & 3 & \dots & n\\ n & n & n & \dots & n\end{pmatrix} & p\begin{pmatrix}1 & 2 & 3 & \dots & n\\ n & n & n & \dots & n-1\end{pmatrix}\\ q\begin{pmatrix}1 & 2 & 3 & \dots & n\\ n & n & n & \dots & n\end{pmatrix} & q\begin{pmatrix}1 & 2 & 3 & \dots & n\\ n & n & n & \dots & n-1\end{pmatrix}\end{bmatrix}$$

$$= \begin{bmatrix} p\begin{pmatrix}1 & 2 & 3 & \dots & n-1\\ n & n & n & \dots & n\end{pmatrix} & p\begin{pmatrix}1 & 2 & 3 & \dots & n-1\\ n & n & n & \dots & n-1\end{pmatrix}\\ q\begin{pmatrix}1 & 2 & 3 & \dots & n-1\\ n & n & n & \dots & n\end{pmatrix} & q\begin{pmatrix}1 & 2 & 3 & \dots & n-1\\ n & n & n & \dots & n-1\end{pmatrix}\end{bmatrix}$$

$$\times \begin{bmatrix} a_{n1} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{n2} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{n3} & 1\\ 1 & 0\end{bmatrix}\dots\begin{bmatrix} a_{nn} & 1\\ 1 & 0\end{bmatrix}.$$

$$= \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$= \begin{bmatrix} a_{11} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{12} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{13} & 1\\ 1 & 0\end{bmatrix}\dots\begin{bmatrix} a_{1n} & 1\\ 1 & 0\end{bmatrix}$$

$$\times \begin{bmatrix} a_{21} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{22} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{23} & 1\\ 1 & 0\end{bmatrix}\dots\begin{bmatrix} a_{3n} & 1\\ 1 & 0\end{bmatrix}$$

$$\times \begin{bmatrix} a_{31} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{32} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{33} & 1\\ 1 & 0\end{bmatrix}\dots\begin{bmatrix} a_{3n} & 1\\ 1 & 0\end{bmatrix}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\times \begin{bmatrix} a_{n1} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{n2} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} a_{n3} & 1\\ 1 & 0\end{bmatrix}\dots\begin{bmatrix} a_{nn} & 1\\ 1 & 0\end{bmatrix}. \qquad (20)$$

Our thanks are due to the University of Patna for the Research Scholarship given to one of us (S. D. Upadhyay) which made this work possible.

DEPARTMENT OF MATHEMATICS,
SCIENCE COLLEGE, PATNA.

References

Milne-Thomson, (1933a), *Calculus of Finite Differences* (*Macmillan*), chap. V.
——— , (1933b), *Proc. Edin. Math. Soc.* (2), 3, 189.

THE ASYMPTOTIC NUMBER OF THREE-DEEP LATIN RECTANGLES

By

S. M. Kerawala

(*Received June 5, 1947*)

A recent paper by Erdös and Kaplansky (1946) establishes the following asymptotic expansion for the number $f(n, k)$ of n by k Latin rectangles:

$$\frac{f(n,k)\exp(kC_2)}{(n!)^k} = 1 - \frac{k(k-1)(k-2)}{6n} + \frac{k(k-1)(k-2)(k^3-3k^2+8k-30)}{72n^2} + \dots,$$

the existence of which was previously conjectured by Jacob (1930). For $k=3$, the expansion reduces to

$$\frac{f(n,3)\,e^3}{(n!)^3} = U_n = 1 - \frac{1}{n} - \frac{1}{2n^2} + \dots,$$

Comparing the value of

$$1 - \frac{1}{n} - \frac{1}{2n^2}$$

with the exact value of U_n given by me (1941) for $n \leq 25$, Erdös and Kaplansky show that the first three terms of their expansion provide numerical results accurate to four figures for $n \geq 20$, whereas MacMahon's (1915) operator or my difference-equation (Kerawala, 1941) or Riordan's combinatory formula (1945) would require heavier computations.

However, the method by which Erdös and Kaplansky derive their general asymptotic formula lacks the power for pushing their series any further. The method, if pursued, may with luck give the coefficient of n^{-3}, but there appears to be no hope of obtaining the coefficients of higher terms by their method. The purpose of the present paper is to derive the higher terms of the asymptotic series when $k=3$.

In my earlier paper (1941), I have shown that U_n satisfies the following difference equation:

$$U_n = \frac{(n-1)(n^2-2n+2)}{n^2(n-2)}U_{(n-1)} + \frac{(n^2-2n+2)}{n^2(n-1)}U_{(n-2)} + \frac{(n^2-2n-2)}{n^2(n-1)(n-2)^2}U_{(n-3)}$$
$$+ \frac{2(n^2-5n+8)}{n^2(n-1)(n-2)^2(n-3)}U_{(n-4)} - \frac{4}{n^2(n-2)^2(n-3)(n-4)}U_{(n-5)}$$

I substitute

$$U_n = 1 + \sum_{r=1}^{\infty}(a_r n^{-r}),$$

assuming the expansion convergent for $n > n_0$. On equating the coefficients o $n^{-2}, n^{-3}, n^{-4}, \ldots$ on both sides and simplifying, I obtain:

$$\begin{aligned}
&1+a_1 &&= 0,\\
&4+3a_1+2a_2 &&= 0,\\
&10+8a_1+6a_2+3a_3 &&= 0,\\
&20+23a_1+17a_2+10a_3+4a_4 &&= 0,\\
&26+68a_1+51a_2+32a_3+15a_4+5a_5 &&= 0,\\
&-32+197a_1+168a_2+104a_3+54a_4+21a_5+6a_6 &&= 0,\\
&-462+520a_1+596a_2+365a_3+193a_4+84a_5+28a_6+7a_7 &&= 0,\\
&-2664+1017a_1+2165a_2+1414a_3+723a_4+330a_5+123a_6+36a_7+8a_8 &&= 0,\\
&-12662-484a_1+7551a_2+5886a_3+2962a_4+1326a_5+528a_6+172a_7+45a_8+9a_9 &&= 0,\\
&-55648-22507a_1+22818a_2+24968a_3+13295a_4+5707a_5+2279a_6+801a_7+232a_8\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +55a_9+10a_{10} &&= 0,\\
&\qquad\text{etc.}\qquad\qquad\text{etc.}\qquad\qquad\text{etc.}
\end{aligned}$$

On solving these, I find

$$\begin{aligned}
1!\,a_1 &= -1, & 6!\,a_6 &= 6961,\\
2!\,a_2 &= -1, & 7!\,a_7 &= 38366,\\
3!\,a_3 &= 2, & 8!\,a_8 &= -1899687,\\
4!\,a_4 &= 49, & 9!\,a_9 &= -133065253,\\
5!\,a_5 &= 629, & 10!\,a_{10} &= -6482111309,\\
&\text{etc.} & \text{etc.} & \quad \text{etc.}
\end{aligned}$$

Hence,

$$\frac{f(n,3)e^3}{(n!)^3} = 1-\frac{1}{n}-\frac{1}{2n^2}+\frac{2}{3!\,n^3}+\frac{49}{4!\,n^4}+\frac{629}{5!\,n^5}+\frac{6961}{6!\,n^6}$$

$$+\frac{38366}{7!\,n^7}-\frac{1899687}{8!\,n^8}-\frac{133065253}{9!\,n^9}-\frac{6482111309}{10!\,n^{10}}-\text{etc.}$$

The first three terms are identical with the three terms of the Erdös-Kaplansky expansion. The series may be pushed further to any desired extent.

DEPARTMENT OF MATHEMATICS,
PRESIDENCY COLLEGE,
CALCUTTA

References

Dulmage, L., (1944), *American Mathematical Monthly*, 51, 586-587.
Erdös P. and Kaplansky, I., (1946), *American Journal of Mathematics*, 68, 230-236.
Jacob, S. M., (1930), *Proc. London Math Soc.*, 31, 329-354.
Kerawala, S. M., (1941), *Bull. Cal. Math. Soc*, 33, 119-127
MacMahon, P. A., (1915), *Combinatory Analysis*, *Vol. I*, Cambridge, 246-263.
Riordan, J., (1944), *American Mathematical Monthly*, 51, 450-452.
——, (1945), *American Mathematical Monthly*, 53, 18-20.

EINIGE DIOPHANTISCHE PROBLEME UND RESULTAT

By

Alfred Moessner

(*Communicated by the Secretary—Received June 23, 1947*)

I. Wir untersuchen die Relation $(-a), (-b), c \overset{2}{=} a, b, (-c)$, wo $a+b=c$.

(1) Besteht die Relation $(-a), (-b), c \overset{2}{=} a, b, (-c)$, dann ist, wenn $s = \frac{a^2+b^2+c^2}{2}$, die Summe

$$(-a+s)^2+(-b+s)^2+(c+s)^2 = (a+s)^2+(b+s)^2+(-c+s)^2$$

durch s ohne Rest teilbar.

Exempel: $(-2), (-5), 7 \overset{2}{=} 2, 5, (-7)$; $s=39$; es ist $(-2+39)^2+(-5+39)^2+(-7+39)^2$ teilbar durch 39.

(2) Besteht die Relation $(-a), (-b), c \overset{2}{=} a, b, (-c)$, dann ist

$$(-a)^2+(-b)^2+c^2 = a^2+b^2+(-c)^2 = 2(a^2+bc) = 2(b^2+ac) = 2(c^2+ab).$$

Setzt man nun $\sqrt{\left\{\frac{a^2+b^2+c^2}{2}\right\}} = k$, dann ist

$$\frac{(-a+k)^2+(-b+k)^2+(c+k)^2}{k^2} = \frac{(a+k)^2+(b+k)^2+(-c+k)^2}{k^2} = 5.$$

Exempel: $(-2), (-3), 5 \overset{2}{=} 2, 3, (-5)$; $k = \sqrt{19}$ und est ergibt

$$\frac{(-2+\sqrt{19})^2+(-3+\sqrt{19})^2+(5+\sqrt{19})^2}{19} = \frac{(2+\sqrt{19})^2+(3+\sqrt{19})^2-(-5+\sqrt{19})^2}{19} = 5.$$

N.B.! Bei (1) und (2) ist vorausgesetzt, dass $a+b=c$ ist.

(3) Ist $a+b=c$ und gilt $(-a), (-b), c \overset{2}{=} a, b, (-c)$ und ist $k = \sqrt{\tfrac{1}{2}(a^2+b^2+c^2)}$, dann gilt

$$\begin{cases} (-a+k), (-b+k), (c+k) \overset{2}{=} (a+k), (b+k), (-c+k) \\ (-a+k).\ (-b+k).\ (c+k) = -(a+k).\ (b+k).\ (-c+k). \end{cases}$$

Exempel: $1+2=3$; also $(-1), (-2), 3 \overset{2}{=} 1, 2, (-3)$; $k = \sqrt{\tfrac{1}{2}(1^2+2^2+3^2)} = \sqrt{7}$, also gilt

$$\begin{cases} (-1+\sqrt{7}), (-2+\sqrt{7}), (3+\sqrt{7}) \overset{2}{=} (1+\sqrt{7}), (2+\sqrt{7}), (-3+\sqrt{7}) \\ (-1+\sqrt{7}).\ (-2+\sqrt{7}).\ (3+\sqrt{7}) = -(1+\sqrt{7}).\ (2+\sqrt{7}).\ (-3+\sqrt{7}). \end{cases}$$

(4) (α) Soll k eine rationale Zahl sein, so müssen a, b, c nicht nur der Kondition $a+b=c$, sondern auch der Bedingung genügen

$$\begin{cases} a^2+b^2+c^2 = 2k^2 \\ a^4+b^4+c^4 = 2k^4, \end{cases}$$

wofür die allgemeine Lösung gilt

$$k = u^2+uv+v^2; \quad a = u^2-v^2; \quad b = 2uv+v^2; \quad c = 2uv+u^2.$$

Exempel: $u = 2$, $v = 1$ ergibt $3, 5, 8 \overset{n}{=} 7, 7$ für $n = 2, 4$. Daraus folgt

$$\begin{cases} (-3+7), (-5+7), (8+7) \overset{2}{=} (3+7), (5+7), (-8+7) \\ (-3+7).(-5+7).(8+7) = -(3+7).(5+7).(-8+7), \end{cases}$$

also die Relation

$$\begin{cases} 4, 2, 15 \overset{2}{=} 10, 12, (-1) \\ 4.2.15 = -10.12.(-1). \end{cases}$$

(β) Gilt die unter (3) angeführte Relation

$$\begin{cases} (-a+k), (-b+k), (c+k) \overset{2}{=} (a+k), (b+k), (-c+k) \\ (-a+k).(-b+k).(c+k) = -(a+k).(b+k).(-c+k), \end{cases}$$

mit der Voraussetzung, dass

$$a, b, c \overset{n}{=} k, k \quad \text{für} \quad n = 2, 4, \quad \text{wo} \quad a+b = c,$$

dann ist

$$\begin{cases} (-a+k) + (-b+k) + (c+k) = (a+k) + (b+k) + (-c+k) = 3k; \\ (-a+k)^2 + (-b+k)^2 + (c+k)^2 = (a+k)^2 + (b+k)^2 + (-c+k)^2 = 5k^2; \\ (-a+k) \,.\, (-b+k) \,.\, (c+k) = (a+k) \,.\, (b+k) \,.\, (-c+k) = a.b.c. \end{cases}$$

II. (1) Wir lösen Relationen, bei welchen die Summe zweier Potenzen gleich ist einer Potenz der Differenz ihrer Basen.

Es gilt $X^3+Y^3 = (X-Y)^4$, wenn wir setzen

$$X = q(8q+1)(64q^2+3); \quad Y = q(8q-1)(64q^2+3).$$

Exempel: $q = 1$ ergibt $603^3+469^3 = (603-469)^4$.

Eine andere Lösung ist gegeben, indem man setzt

$$X = (r+1)(2r+1)(r^2+r+1); \quad Y = r(2r+1)(r^2+r+1).$$

Exempel: $r = 2$ ergibt $105^3+70^3 = (105-70)^4$

Eine Lösung der allgemeineren Gleichung

$$X^n+Y^n = (X-Y)^{n+1}$$

ist gegeben, wenn man setzt

$$X = (p+1)a; \quad Y = pa; \quad \text{wobei } a = (p+1)^n+p^n.$$

Exempel: $p=3$, $n=4$ ergibt, da $a=4^4+3^4=337$ ist, das Resultat :

$$1348^4+1011^4 = (1348-1011)^5.$$

Die Relation $A^2+R^2 = (A-B)^3$ hat die Spezielle Lösung

$$A = (2k^2+2k+1)(k+1); \quad B = k(2k^2+2k+1).$$

Exempel: $k=3$ ergibt $100^2+75^2 = (100-75)$.

Wir lösen ähnliche Relationen. Es ist $X+Y = X^2+Y^2$, wenn man setzt

$$X = \frac{a}{a^2+1}\cdot(a+1); \quad Y = \frac{a}{a^2+1}\cdot(a-1).$$

Exempel: $a=3$ ergibt die Lösung

$$\frac{6}{5}+\frac{3}{5} = \left\{\frac{6}{5}\right\}^2+\left\{\frac{3}{5}\right\}^2 = \frac{9}{5}.$$

Die Relation

$$\begin{cases} A,\ B \overset{2}{=} C,\ D,\ E \\ A+B = A^3+B^3 \end{cases}$$

lösen wir, indem wir setzen

$$A = \frac{m^2-2m}{m^2-m+1}; \quad B = \frac{m^2-1}{m^2-m+1}; \quad C = s-B, \quad D = s-A; \quad E = s;$$

wobei $s=\frac{2}{3}.(A+B)$.

Exempel: $m=4$ ergibt, da dann

$$s = \frac{2}{3}.\left\{\frac{8}{13}+\frac{15}{13}\right\} = \frac{46}{39},$$

die Lösung

$$\begin{cases} \frac{8}{13},\ \frac{15}{13} \overset{2}{=} \frac{1}{39},\ \frac{22}{39},\ \frac{46}{39} \\ \frac{8}{13}+\frac{15}{13} = \left\{\frac{8}{13}\right\}^3+\left\{\frac{15}{13}\right\}^3. \end{cases}$$

Bei der Relation $X^3+Y^3 = (X+Y)^2$ ergeben sich für Y zwei Werte, und zwar ist

$$X = \frac{p^2-q^2}{p^2-4pq+7q^2}; \quad Y_1 = \frac{2p^2-6pq+4q^2}{p^2-4pq+7q^2}; \quad Y_2 = \frac{2(pq+q^2)}{p^2-4pq+7q^2}.$$

Exempel: $p=4$, $q=1$ ergibt $X = \frac{15}{7}$, $Y_1 = \frac{12}{7}$, $Y_2 = \frac{10}{7}$, also die Lösung

$$\left\{\frac{15}{7}\right\}^3+\left\{\frac{12}{7}\right\}^3 = \left\{\frac{15}{7}+\frac{12}{7}\right\}^2 \quad \text{und} \quad \left\{\frac{15}{7}\right\}^3+\left\{\frac{10}{7}\right\}^3 = \left\{\frac{15}{7}+\frac{10}{7}\right\}^2$$

Eine Lösung zu $C-D = C^2-D^3$ ist gegeben, wenn wir setzen

$$C = \frac{2a+1}{a^2+a+1}, \quad D = \frac{a^2+1}{a^2+a+1}.$$

Exempel: $a=2$ ergibt

$$\frac{5}{7}-\frac{3}{7}=\left\{\frac{5}{7}\right\}^3-\left\{\frac{3}{7}\right\}^3.$$

Besteht die Relation $e+f=g$, dann gilt auch

$$\left[e+f-\frac{ef}{g}\right]^2=e^2+f^2+\left[-\frac{ef}{g}\right]^2.$$

Exempel: Da $3+4=7$, so gilt

$$\left[3+4-\frac{12}{7}\right]^2=3^2+4^2+\left[-\frac{12}{7}\right]^2.$$

Es ist $k^4=[x+y+(-z)]^2=x^2+y^2+(-z)^2$, wenn man setzt

$$x=\frac{2k^2q(q+1)}{3q^2+1}\ ;\qquad y=\frac{2k^2q(q-1)}{3q^2+1}\ ;\qquad z=\frac{k^2(1-q^2)}{3q^2+1}.$$

Exempel: $k=7$, $q=3$ ergibt das Resultat

$$7^4=[42+21+(-14)]^2=42^2+21^2+(-14)^2.$$

(2) Ich möchte einige verwandte Probleme angeben, deren Lösungen mir noch nicht bekannt sind und zwar (α) $X^n+Y^n=(X-Y)^n$; (β) $(A+B)^p=A^{p+1}+B^{p+1}$; deren Lösung wohl nur mit komplexen Zahlen möglich ist; (γ) $A^m+B^m=(A-B)^{m-1}$; (δ) $G^v.H^v=(G+H)^v$. Hiefür ist für den Fall $v=2$ eine numerische Lösung

$$3^2.\left[\frac{3}{2}\right]^2=\left[3+\frac{3}{2}\right]^2;\quad (\varepsilon)\ G^v.H^v=(G-H)^v;$$

eine numerische Lösung hiefür ist

$$3^2.\left[-\frac{3}{4}\right]^2=\left[3-\frac{3}{4}\right]^2.$$

(3) Das Gleichungs system

$$\left\{\begin{array}{llll}
A^4+B^4+C^4 & +a+b+c & = D^2+E^2+F^2 & +d+e+f \\
A^8+B^8+C^8 & +a^2+b^2+c^2 & = D^4+E^4+F^4 & +d^2+e^2+f^2 \\
A^{12}+B^{12}+C^{12} & +a^3+b^3+c^3 & = D^6+E^6+F^6 & +d^3+e^3+f^3 \\
A^4+B^4+C^4 & & = D^2+E^2+F^2 & \\
 & a+b+c & = & d+e+f \\
A^8+B^8+C^8 & & = D^4+E^4+F^4 & \\
 & a^2+b^2+c^2 & = & d^2+e^2+f^2 \\
A^2+B^2=C^2 & & \quad D^2+E^2=F, &
\end{array}\right.$$

kann in ganzen positiven Zahlen auf folgende Weise gelöst werden.

Wir lösen zunächst die pythagoräische Gleichung $A^2+B^2=C^2$ nach den Formeln von Brahmagupta $A^2=(x^2-y^2)$, $B^2=(2xy)^2$, $C^2=(x^2+y^2)^2$. Haben wir so A, B, C

bestimmt, dann bestimmen wir F, D, E und zwar ist $D = B^2 - A^2 - AB$, $E = 2AB$. $F = D + E$. Damit können wir die Identität

$$\begin{cases} A^4 + B^4 + C^4 = D^2 + E^2 + F^2 \\ A^8 + B^8 + C^8 = D^4 + E^4 + F^4 \end{cases}$$

bestimmen.

Diese Identität ist identisch mit der Relation

$$(A^4), (B^4), (C^4) \overset{2}{=} (D^2), (E^2), (F^2).$$

Gilt die Identität

$$G_1, G_2, \ldots, G_r \overset{n}{=} K_1, K_2, \ldots, K_r$$

dann gilt für beliebiges h die Relation

$$G_1, G_2, \ldots, G_r, K_{1+h}, K_{2+h}, \ldots K_{r+h} \overset{3}{=} K_1, K_2, \ldots, K_r, G_{1+h}, G_{2+h}, \ldots, G_{r+h}.$$

Wir wenden dieses Theorem an auf die Identität

$$(A^4), (B^4), (C^4) \overset{2}{=} (D^2), (E^2), (F^2)$$

und erhalten damit Lösungen der Identität

$$(A^4), (B^4), (C^4), a, b, c \overset{3}{=} (D^2), (E^2), (F^2), d, e, f.$$

Damit sind auch die drei ersten Gleichungen des vorgelegten Gleichungssystems gelöst

Exempel: $x=2$ $y=1$ ergibt

$$3^2 + 4^2 = 5^2, \quad D = 4^2 - 3^2 - 3.4 = (-5), \quad E = 2.3.4 = 24, \quad F = (-5) + 24 = 19.$$

Wir können das negative Vorzeichen bei (-5) unberücksichtigt lassen, da es sich nur um gerade Potenzen handelt, können also schreiben

$$\begin{cases} 3^4 + 4^4 + 5^4 = 5^2 + 19^2 + 24^2 \\ 3^8 + 4^8 + 5^8 = 5^4 + 19^4 + 24^4. \end{cases}$$

Setzen wir $h=1$, so ergeben sich die Werte

$$a = 5^2 + 1 = 26, \quad b = 19^2 + 1 = 361, \quad c = 24^2 + 1 = 577, \quad d = 3^4 + 1 = 82,$$
$$e = 4^4 + 1 = 257, \quad f = 5^4 + 1 = 626.$$

(4) Zu der simultanen Identität $A, B, C, D. \overset{n}{=} E, F, F, G$ $(n = 1, 2, 4)$ fand ich die Lösung

$$\begin{aligned} A &= a.b.(2a+b)^2; & E &= a.(a+ab+b) + b(a+b)(a+2b); \\ B &= (a^2-b^2)^2; & F &= b.(a-b).(2a+b); \\ C &= b^2(2a+b)^2; & G &= (a+2b).(a+ab+b). \\ D &= (a^2-b^2)(a+b)(a+2b); & & \end{aligned}$$

Hiebei ist

$$A - F = G - B; \qquad D - F = E - C; \qquad C.(D + 2B + E) = A^2;$$

$$\frac{D+2B}{D-B} = \frac{A}{C} = \frac{a}{b} = \frac{D+2B+E}{A}; \qquad \frac{D+2B+E}{C} = \frac{a}{b}.$$

Exempel: $a=2$, $b=1$ ergibt das Resultat $50, 9, 25, 36 \overset{n}{=} 46, 15, 15, 44$ $(n=1, 2, 4)$.

(5) Zum Schlusse führe ich noch die mir bis jetzt bekannten Lösungen mit Zahlen 1000 zur Gleichung.

$M, P, Q \overset{n}{=} U, V, W$ für $n = 1, 5$ an:

$$(\alpha) \qquad 49, 75, 107 \overset{n}{=} 39, 92, 100 \quad (n = 1, 5);$$

$$(\beta) \qquad 293, 754, 911 \overset{n}{=} 498, 501, 959 \quad (n = 1, 5);$$

$$(\gamma) \qquad (-56), 417, 618 \overset{n}{=} 161, 184, 634 \text{ für } (n = 1, 5);$$

$$(\delta) \qquad 123, 389, 442 \overset{n}{=} 208, 272, 474 \quad (d = 1, 5).$$

GUNZENHAUSEN (GERMANY—BAYERN)

PROPERTIES OF THREE NUMBERS IN A-G.P.*

By

Abdur Rahman Nasir

(*Communicated by the Secretary—Received April 8, 1947*)

1. In this note I.R. stands for "Indicator of Ratio" which means the common ratio of the component G.P. of an A-G.P. An A-G.P. having R as I.R. is denoted by "A-G.P.(R)" or simply by (R).

We shall start with a lemma which can be easily proved.

Lemma. *If a, b, c are any three consecutive terms of (R), then aR^2, bR, c are in A.P.*

2. **Theorem I.** *If f, g, h are in A-G.P.(r), then they are also in A-G.P.(h/fr), provided $f \neq 0$ and $h \neq 0$.*

The given numbers can be written as

$$f, \quad \frac{fgr}{h}\left(\frac{h}{fr}\right), \quad \frac{f^2r^3}{h}\left(\frac{h}{fr}\right)^2, \tag{1}$$

which can be obtained by multiplying the corresponding terms of the following sets

$$1, \quad \left(\frac{h}{fr}\right), \quad \left(\frac{h}{fr}\right)^2, \tag{2}$$

$$f, \quad \frac{fgr}{h}, \quad \frac{f^2r^3}{h}. \tag{3}$$

Clearly (2) is a G.P.

Again, h, gr, fr^2 are in A.P.(Lemma).

Multiplying by the same number f/h, we, at once, see that (3) is an A.P.

Hence (1) is an A-G.P.(h/fr), which proves the theorem.

3. **Theorem II.** *Three consecutive terms of an A-G.P. cannot be common to more than two A-G.P.'s with distinct I.R.'s.*

Let l, m, n be three consecutive terms of an A-G.P.(t). Then lt^2, mt, n are in A.P.(Lemma).

Therefore

$$mt - lt^2 = n - mt.$$

or

$$lt^2 - 2mt + n = 0, \tag{4}$$

which determines the value of t.

It is clear that the given terms are common to A-G.P's whose I.R's are the roots of the equation (4).

---

* Arithmetico-Geometrical Progression.

But the equation (4) is quadratic in t and thus cannot have more than two distinct roots.

It therefore follows that the given three consecutive terms cannot be exhibited as A-G.P. in more than two ways with distinct I.R's.

Hence the theorem.

4. **Theorem III**. *Two A-G.P's cannot have more than three consecutive terms common without becoming identical.*

Let p, q, r, s be four consecutive terms of an A-G.P.

Suppose that p, q, r are common to two A-G.P's, (R_1) and (R_2). We shall show that s belongs to either (R_1) or (R_2) and cannot belong to both (R_1) and (R_2).

By Theorem II, R_1 and R_2 are the roots of the equation

$$px^2-2qx+r=0. \tag{5}$$

If possible, let s belong to (R_1) and (R_2). Then (R_1) and (R_2) have also the three consecutive terms q, r, s in common.

Hence R_1 and R_2 are also the roots of the equation

$$qx^2-2rx+s=0. \tag{6}$$

Since equations (5) and (6) have two roots equal, we have

$$p/q=q/r=r/s,$$

that is, p, q, r, s are in continued proportion—a contradiction to the hypothesis that p, q, r, s form four consecutive terms of an A-G.P.

Therefore s cannot belong to both (R_1) and (R_2) without making $(R_1) \equiv (R_2)$.

Hence s belongs to either (R_1) or (R_2).

This proves our theorem.

Theorem IV. *If a, b, c, d are four consecutive terms of an A-G.P. then*

$$(bc-ad)^2=4(b^2-ac)(c^2-bd).$$

The result readily follows in view of the fact that the two quadratics

$$at^2-2bt+c=0,$$

$$bt^2-2ct+d=0,$$

must have one root in common.

5. If A, B, C are any three consecutive terms of an A-G.P.(R), from our lemma we infer that these can be put as A, $(A+D)R$, $(A+2D)R^2$. D is called "Indicator of Difference" (written I.D. for short) corresponding to R relative to the given three terms.

Theorem V. *Ratio of I.R's of two A-G.P.'s that have the given three consecutive terms in common is equal to the negative reciprocal of the ratio of the corresponding I.D's relative to the given three terms.*

Let x, y, z be common to A-G.P.'s, (T_1) and (T_2). If D_1 and D_2 be I.D's corresponding to T_1 and T_2 respectively, we have

$$y = (x+D_1)T_1 = (x+D_2)T_2.$$

Therefore

$$D_1T_1 = y-xT_1, \qquad D_2T_2 = y-xT_2.$$

Adding these we have

$$D_1T_1+D_2T_2 = 2y-x(T_1+T_2).$$

But by Theorem II,

$$T_1+T_2 = 2y/x.$$

Therefore

$$D_1T_1+D_2T_2 = 0.$$

Hence the theorem.

Cor. 1. If T_1 and T_3 have the same sign, D_1 and D_2 must have opposite signs and *vice versa*.

Cor. 2. $D_1T_1^2+D_1T_1^2 = (2/x)(zx-y^2)$.

6. I am not aware of any text book on Algebra in which the above properties of this series have been discussed and for this reason I am led to believe that the results are new.

I stop here and hope to return to the subject later.

QUADIAN, W. PUNJAB,
PAKISTAN.

Reference

Abdur Rahman Nasir, (1947), *Current Science*, (1), 16, 19.

ASYMPTOTIC SOLUTION OF THE "PROBLÈME DES MÉNAGES"

By

S. M. Kerawala

(*Received June 28, 1947*)

The "problème des ménages" consists in finding the number of ways in which n married couples may be seated at a round table, men alternating with women, so that no wife sits next to her own husband.

Let the number of ways be set equal to $2(n!)a(n)$. Several solutions have been given before by expressing $a(n)$ in various ways. Mac Mahon (1915), for instance, gives the following operational formula for $a(n)$:

$$a(n) = (3!/n!)^n[D_{a+b+c+}\dots(cde\dots)(ade\dots)(abe\dots)\dots]_{a=1} \tag{1}$$

Moreau (Lucas, 1891) gives a difference equation for $a(n)$ equivalent to

$$(n-2)a(n) = n(n-2)a(n-1)+na(n-2)-4(-1)^n, \tag{2}$$

and then calculates $a(n)$ for $n \leq 20$.

Recently, however, more explicit formulæ have been obtained for $a(n)$. Touchard (1934) expresses $a(n)$ in terms of definite integrals, and Schöbe (1943) shows that

$$a(n+1) = 2(-1)^{n+1}+\frac{(n+1)(-1)^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}h_k^2, \tag{3}$$

where h_k is Whitworth's "sub-factorial" k or

$$h_k = k!\sum_{r=0}^{k}(-1)^r/(r!).$$

Schöbe also establishes that

$$a(n)/n! \sim 1/e^2 \tag{4}$$

Very recently, Kaplansky (1945) by an interesting application of Cayley and Sylvester's principle of "cross-classification" showed that

$$a(n) = 2(-1)^n+(-1)^{n-1}n\sum_{r=0}^{n-1}(-1)^r\binom{n+r}{2r+1}r! \tag{5}$$

It is intended here to establish an asymptotic series for $a(n)$, assuming (2) and (4).

I

To investigate a particular solution of the difference equation (2), I set

$$a(n) = \sum_r c_r n^{-r},$$

assuming convergence for all $n \geq n_0$, and find

$$c_0 = 0, \quad c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = 0,$$

$$c_2 = 4(-1)^n, \quad c_4 = -5c_2, \quad c_6 = 31c_2, \quad c_8 = -125c_2, \quad c_{10} = -1769c_2, \text{ etc.}$$

Hence the particular solution runs

$$4(-1)^n\left[\frac{1}{n^2}-\frac{5}{n^4}+\frac{31}{n^6}-\frac{125}{n^8}-\frac{1769}{n^{10}}\cdots+\cdots\right].$$

Next, I seek a solution of the homogeneous equation

$$(n-2)a(n)-n(n-2)a(n-1)-na(n-2) = 0 \tag{6}$$

such that

$$a(n)/(n!) \sim e^{-2}.$$

Writing

$$a(n) = \frac{n!}{e^2}\left[1+\sum_{r=1}^{\infty} d_r n^{-r}\right]$$

in (6) and assuming the infinite series to converge for $n > n_1$, I find

$1!\ d_1 = -1,$ $\quad 6!\ d_6 = 1501,$
$2!\ d_2 = -1,$ $\quad 7!\ d_7 = -31354,$
$3!\ d_3 = 2,$ $\quad 8!\ d_8 = -1451967,$
$4!\ d_4 = 37,$ $\quad 9!\ d_9 = -39284461,$
$5!\ d_5 = 329,$ $\quad 10!\ d_{10} = -737652869,$ etc.

Hence the asymptotic value of $a(n)$ is given by

$$\begin{aligned}a(n) = \frac{n!}{e^2}\Big[1-\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\frac{37}{24n^4}+\frac{329}{120n^5}+\frac{1501}{720n^6}\\ -\frac{15677}{2520n^7}-\frac{1451967}{40320n^8}-\frac{39284461}{362880n^9}-\frac{737652869}{3628800n^{10}}-\cdots+\cdots\Big]\\ +4(-1)^n\left[\frac{1}{n^2}-\frac{5}{n^4}+\frac{31}{n^6}-\frac{125}{n^8}-\frac{1769}{n^{10}}-\cdots+\cdots\right]\end{aligned} \tag{7}$$

I believe that for large values of n, the above provides a more explicit solution of the "problème des ménages" than any previously given.

II

As Lucas 1891 remarks, $a(n)$ is the number of permutations of $1, 2, \ldots n$ discordant with each of the two permutations

$$1,\ 2,\ 3,\ 4,\ \ldots\ (n-1),\ n$$
$$2,\ 3,\ 4,\ 5,\ \ldots\quad n,\quad 1.$$

If the two lines be replaced by any two discordant permutations of $1, 2, \ldots n$, we get the problem of completing the third line in a three-deep Latin rectangle, of which the first two lines are fixed. The asymptotic number of solutions in this case was recently found by Erdös and Kaplansky (1946) to be

$$\frac{n!}{e^3}\left[1-\frac{1}{n}-\frac{1}{2n^2}+\cdots\right];$$

but the series has been extended by me (1947) as follows:

$$\frac{n!}{e^3}\left[1-\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\frac{49}{24n^4}+\frac{629}{120n^5}+\frac{6961}{720n^6}+\frac{19183}{2520n^7}\right.$$
$$\left.-\frac{633229}{13440n^8}-\frac{138065258}{362880n^9}-\frac{6482111309}{3628800n^{10}}-\cdots+\cdots\right].$$

It is instructive to compare this last result with (7). The two asymptotic expansions are identical as far as the fourth term, but differ from the fifth term onwards. This throws interesting light on some conjectures of Erdös and Kaplansky about the asymptotic number of the general n by k Latin rectangle. I hope to examine the implications in a subsequent paper.

PRESIDENCY COLLEGE,
CALCUTTA

References

Erdös, P. and Kaplansky, I., (1946), *Amer. J. Math.*, **68**, 230-236.
Kaplansky I., (1945), *Bull. Amer. Math. Soc*, **49**, 784-785.
Kerawala, S. M., (1947), *Bull. Cal. Math. Soc*, **39**, 71-72.
Lucas, (1891), *Théorie des nombres*, Gauthier-Villars, Paris, 491 495.
Mac Mahon, P. A., (1915), Vol. 1, *Combinatory Analysis*, Cambridge, 253-257.
Schöbe, W., (1943), *Math. Z.* **48**, 781-784.
Touchard, (1934), *C. R. Acad. Sci.*, Paris, **198**, 631-633.

PONCELET PORISM IN TWO CIRCLES

By

S. M. Kerawala

(Received May 22, 1947)

INTRODUCTION

The general problem of investigating the condition that an n-agon inscribed in a given conic S should circumscribe any other given conic s has in some form or the other received the attention of a galaxy of prominent mathematicians including Euler, Steiner [16] and Cayley [1], [2], [3]. The general porisms in terms of the invariants and covariants of S and s rapidly become unmanageable and do not appear to have claimed any great sustained interest. It is otherwise when S and s are considered to be respectively reduced to homographically equivalent circles of radii R and r with centres a distance d apart. Weill [18], Titchmarsh [17], Rosanes and Pasch [15], Chaundy [5], [6] and numerous others have delved by such methods into the secrets of a circle-pair, but the palm appears to me to have been carried away by Chaundy, who in his two papers has veritably unearthed a priceless treasure.

In his first paper, Chaundy shows that the porism of a $2n$-agon, when purged of all irrelevant factors, splits into two factors, one simplicity itself in comparision with the other. In conformity with Chaundy's nomenclature and subsequent notation, the bulkier factor will be said to correspond to the porism of the "inseparable" case and will be denoted by $K(2n)$. Of the more docile factor, corresponding to the "separable" case, there always happen to be two forms. If $2n = 2^r.(2m+1)$, these two conditions can be associated with the two forms for the porism of a $(2m+1)$-agon, and in conformity with what follows, these will respectively be denoted by $C_1(2n)$ and $C_2(2n)$, or simply by $C(2n)$ when either may be implied. When all redundant factors have been removed, two forms occur of a porism of a $(2n+1)$-agon, one corresponding to the case when s is the ex-circle and the other to the case when s is the in-circle. These will be denoted respectively by $C_1(2n+1)$ and $C_2(2n+1)$, the suffix being dropped when either may be implied. Chaundy proves that to derive $C(4n+2)$ from $C(2n+1)$, an interchange of R and d suffices. Using elliptic functions throughout and converting odd porisms into even by interchanging R and d, he exhibits in their smartest attire the following porisms: $C(3)$, $C(4)$, $C(5)$, $C(6)$, $C(7)$, $C(8)$, $C(9)$, $C(10)$, $C(12)$, $C(14)$, $C(16)$, $C(18)$ and $C(20)$. $C(15)$ and $C(30)$ which also occur are in company of one inferior porism each. This is a weakness to which the great Steiner also succumbed when he held up his $C(5)$ for display, and the error has been pointed out by Chaundy. In fairness, however, to Chaundy, it must be pointed out that the errors in $C(15)$ and $C(30)$ have been rectified in the second paper, where he gives in much less explicit forms $C(15)$, $C(21)$, $C(33)$ and $C(35)$. The irrational representatives of these have not been given in terms of R, r and d—what was Chaundy's objective in the first paper.

Towards the end of his work, Chaundy obtains $C(11)$, $C(22)$, $K(22)$ and $C(44)$. All these expressions are erroneous, as will be shown subsequently, Chaundy concludes his work with the remark: "I have calculated $K(44)$, but it is unprintable." However, as $K(44)$ could only have been calculated from $C(11)$ as a starting-point, the same remark applies to it as to $C(11)$. Chaundy also claims to have scored a century by evaluating $C(112)$. But he complains that the task was a heart-rending one and the final result unprintable.

The purpose of the present investigation is to establish as many porisms as possible in a systematic manner without any use of elliptic functions and by the comparatively elementary methods of the theory of a two-two correspondence. The total number of porisms that can be simply and elegantly derived by these methods transcends the wildest hopes with which I set out upon the task. Many of the results given here, I believe, go far beyond the total ground covered by all the previous writers on the subject. A variety of methods have been developed by the previous writers, but these suffered from the defect that while some yielded the porisms in irrational forms, others yielded them in their rational forms, and the task of transforming the one form into the other was in general a heart-breaking one. The result has been that mathematicians have regarded some one form with greater favour than the other. Chaundy, for example, is all for the irrational form, and he does not appear to be too pleased with the rational porism of an enneagon obtained by Richelot [14]. This defect may now be considered to be removed by the present paper, as the passage from one form to the other may be effected with the same ease with which one may switch on or off an electric light. Another feature, which I believe deserves mention, is the discovery by tentative methods of the general replicating law in its explicit form. Chaundy sets up explicit machinery for duplication alone, but the general approach lies concealed in the formulae for inseparable cases alone, and as Chaundy hardly displays any great interest in these particular cases, he appears to have missed this treasure.

It will be found, however, that elliptic functions do occur towards the end of the paper. But it must be mentioned that they have nothing to do with the main purpose of the paper, and the idea in bringing them in is only to give a completeness to the whole theory, which it would otherwise lack. I must confess, however, that I am deeply indebted for many of the ideas to the works of Weill [18], Clifford [6], [7], [8], Darboux [9], [10] and Titchmarsh [17] on the theory of a two-two correspondence, with which the problem is inextricably bound. But my sincerest thanks reach out to Chaundy whose two papers stood out as guiding stars, whenever I was about to despair.

1

As already mentioned, R will denote the radius of the circum circle S, r the radius of the in-or ex-circle s and d the distance separating their centres. Let

$$2u^2 = R-d+r, \qquad 2v^2 = R+d+r, \qquad w^2 = R$$

and

$$a^{-1} = R+d, \qquad b^{-1} = R-d, \qquad c^{-1} = r.$$

If the chord joining the points $A_i(R\cos\theta_i,\ R\sin\theta_i)$ and $A_{i+1}(R\cos\theta_{i+1},\ R\sin\theta_{i+1})$ on the circle $S \equiv x^2+y^2-R^2=0$ touches the circle $s \equiv (x-d)^2+y^2-r^2=0$, we must have

$$R\cos\tfrac{1}{2}(\theta_{i+1}-\theta_i)-d\cos\tfrac{1}{2}(\theta_{i+1}+\theta_i)=\pm r,$$

the sign to be taken positive if s be the in-circle, negative otherwise. In what follows, we shall choose the positive sign throughout and write $-r$ for r or $-c$ for c to derive $C_1(n)$ from $C_2(n)$ or vice versa. On writing $t_i=\tan\frac{1}{2}\theta_i$, the condition of contact may be rewritten as

$$u^2(t_i+t_{i+1})^2+v^2(1-t_it_{i+1})^2=w^2(1+t_it_{i+1})^2. \tag{1.1}$$

This is the quadratic involution connecting the points A_i, A_{i+1}. Assuming the poristic nature of the problem, no loss of generality occurs by setting $\theta_0=0$, giving $t_0=0$ and $t_1^2u^2=w^2-v^2$. Choosing the positive sign for t_1, we may derive from it two values for t_2, one of which will be t_0. Rejecting the latter value, we get a unique value for t_2. If now the values of $t_0, t_1, t_2, \ldots t_k$ be unique the quadratic involution will determine two values for t_{k+1}, one of which will be t_{k-1}. Rejecting the latter value, t_{k+1} is uniquely determined. This establishes the induction, leading to unique values for each of $t_0, t_1, t_2, \ldots t_i$ for all integral values of i.

The condition that $A_{i-1}A_i$ should touch s is represented by

$$u^2(t_i+t_{i-1})^2+v^2(1-t_it_{i-1})^2=w^2(1+t_it_{i-1})^2 \tag{1.2}$$

From (1.1) and (1.2) it follows that

$$t_{i+1}+t_{i-1}=\frac{-u^2+v^2+w^2}{u^2}\cdot\frac{2t_i}{1-t_i^2t_1^2} \tag{1.31}$$

and

$$t_{i+1}t_{i-1}=\frac{t_i^2-t_1^2}{1-t_i^2t_1^2} \tag{1.32}$$

so that

$$\frac{t_{i+1}+t_{i-1}}{1-t_{i+1}t_{i-1}}=\frac{-u^2+v^2+w^2}{u^2-v^2+w^2}\cdot\frac{2t_i}{1-t_i^2}. \tag{1.33}$$

If we now eliminate t_i between (1.1) and (1.2), we obtain

$$a^2(t_{i+1}+t_{i-1})^2+b^2(1-t_{i-1}t_{i+1})^2=c^2(1+t_{i-1}t_{i+1})^2. \tag{1.4}$$

From the similarity of (1.1) and (1.4), it follows that replacing u, v, w by a, b, c in the three equations (1.3) would lead to

$$t_{i+2}+t_{i-2}=\frac{-a^2+b^2+c^2}{a^2}\cdot\frac{2t_i}{1-t_i^2t_2^2} \tag{1.51}$$

$$t_{i+2}t_{i-2}=\frac{t_i^2-t_2^2}{1-t_i^2t_2^2} \tag{1.52}$$

and

$$\frac{t_{i+2}+t_{i-2}}{1-t_{i+2}t_{i-2}}=\frac{2t_i}{1-t_i^2}\cdot\frac{-a^2+b^2+c^2}{a^2-b^2+c^2}. \tag{1.53}$$

From a comparison of (1.1) and (1.4), it follows further that to convert t_i into t_{2i}, any one of the following equivalent transformations would suffice:

$$u^2 \to a^2, \qquad v^2 \to b^2, \qquad w^2 \to c^2;$$

$$u^2 \to (-u^2+v^2+w^2)^{-2}, \qquad v^2 \to (u^2-v^2+w^2)^{-2}, \qquad w^2 \to (u^2+v^2-w^2)^{-2};$$

$$a \to (-a^2+b^2+c^2)^{-1}, \qquad b \to (a^2-b^2+c^2)^{-1}, \qquad c \to (a^2+b^2-c^2)^{-1};$$

$$R \to c^2, \qquad r \to (a^2+b^2-c^2), \qquad d \to (b^2-a^2).$$

This is the general duplicating transformation which we denote by T_2. If T_2 operates on any homogeneous function of a, b, c of degree i, it is converted into another homogeneous function of a, b, c of degree $4i$. The form $u^2 \to a^2$, $v^2 \to b^2$, $w^2 \to c^2$, however, requires care in operation. If the porism on which it operates is a rational function of u^2, v^2, w^2, duplication is effected without fail. But if the porism is in its simplest irrational form, which is the form preferred by Chaundy, duplication does not always take place on applying the transformation. Four cases may be distinguished:

$$T_0 \equiv u \to a, \qquad v \to b, \qquad w \to c$$

$$T_0' \equiv u \to -a, \qquad v \to b, \qquad w \to c$$

$$T_0'' \equiv u \to a, \qquad v \to -b, \qquad w \to c$$

$$T_0''' \equiv u \to a, \qquad v \to b, \qquad w \to -c.$$

In practice, it is readily found that T_0 only rationalizes the porism and does nothing else; T_0' rationalizes, interchanges R and d and converts r into $-r$; T_0'' rationalizes and interchanges R and d; and finally that T_0''' rationalizes and changes r into $-r$. These transformations will prove their worth in the subsequent discussion.

Repeating on (1.4) the operation by which (9.4) was derived from (1.1) and by merely using T_2, we obtain

$$(-a^2+b^2+c^2)^{-2}(t_{i+2}+t_{i-2})^2+(a^2-b^2+c^2)^{-2}(1-t_{i+2}t_{i-2})^2 = (a^2+b^2-c^2)^{-2}(1+t_{i+2}t_{i-2})^2 \quad (1.6)$$

from which follows

$$t_{i+4}+t_{i-4} = \frac{2t_i}{1-t_4{}^2t_i{}^2}\cdot\frac{\{(-a^2+b^2+c^2)^4-16b^2c^2(b^2-a^2)(c^2-a^2)\}}{(a^2-b^2+c^2)^2(a^2+b^2-c^2)^2} \tag{1.71}$$

$$t_{i+4}.t_{i-4} = \frac{t_i{}^2-t_4{}^2}{1-t_i{}^2t_4{}^2} \tag{1.72}$$

and

$$\frac{t_{i+4}+t_{i-4}}{1-t_{i+4}t_{i-4}} = \frac{(-a^2+b^2+c^2)^2-16b^2c^2(b^2-a^2)(c^2-a^2)}{(a^2-b^2+c^2)^2-16c^2a^2(a^2-b^2)(c^2-b^2)}\cdot\frac{2t_i}{1-t_i{}^2}. \tag{1.73}$$

It follows that the transformation which converts t_r into t_{4r} is T_2 T_2 or $T_2{}^2$ or T_4 equivalent to

$$a \to (-3a^4+b^4+c^4-2b^2c^2+2c^2a^2+2a^2b^2)^{-1}, \text{ etc.}$$

We next seek to eliminate t_{i+1} between

$$u^2(t_i+t_{i+1})^2+v^2(1-t_it_{i+1})^2 = w^2(1+t_it_{i+1})^2$$

and

$$a^2(t_{i+1}+t_{i+3})^2+b^2(1-t_{i+1}t_{i+3})^2 = c^2(1+t_{i+1}t_{i+3})^2,$$

reaching (after wading through a mass of algebra) the objective in the form

$$\{u^2(t_i+t_{i+3})^2+v^2(1-t_it_{i+3})^2-w^2(1+t_it_{i+3})^2\}\left\{\frac{a(b+c)(t_i+t_{i+3})^2}{(-a+b+c)^2}+\frac{b(c+a)(1-t_it_{i+3})^2}{(a-b+c)^2}-\frac{c(a+b)(1+t_it_{i+3})^2}{(a+b-c)^2}\right\}=0.$$

As the first factor would imply a retracing in the path $A_0\,A_1\,A_2\,.\,.\,.$, we reject it. The required eliminant may be written as

$$\frac{a(b+c)(t_i+t_{i+3})^2}{(-a+b+c)^2}+\frac{b(c+a)(1-t_it_{i+3})^2}{(a-b+c)^2}=\frac{c(a+b)(1+t_it_{i+3})^2}{(a+b-c)^2}. \qquad (1.8)$$

It follows that the triplicating transformation T_3 converting t_r into t_{3r} is given by any one of the following equivalent forms:

$$u^2 \to \frac{u^2}{(-a+b+c)^2},\ \text{etc.,} \qquad u^2 \to \frac{a(b+c)}{(-a+b+c)^2},\ \text{etc.,}$$

$$a \to a(-3a^4+b^4+c^4-2b^2c^2+2c^2a^2+2a^2b^2)^{-1},\ \text{etc.}$$

From the involution (1.8), it follows that

$$t_{i-3}+t_{i+3}=\frac{\dfrac{-a(b+c)}{(-a+b+c)^2}+\dfrac{b(c+a)}{(a-b+c)^2}+\dfrac{c(a+b)}{(a+b-c)^2}}{\dfrac{a(b+c)}{(-a+b+c)^2}}\cdot\frac{2t_i}{1-t_i^2t_3^2} \quad \ldots \quad (1.91)$$

$$t_{i-3}\,.\,t_{i+3}=\frac{t_i^2-t_3^2}{1-t_i^2\,t_3^2}, \quad \ldots \quad (1.92)$$

and

$$\frac{t_{i-3}+t_{i+3}}{1-t_{i-3}\,t_{i+3}}=\frac{2t_i}{1-t_i^2}\cdot\frac{\dfrac{-a(b+c)}{(-a+b+c)^2}+\dfrac{b(c+a)}{(a-b+c)^2}+\dfrac{c(a+b)}{(a+b-c)^2}}{\dfrac{a(b+c)}{(-a+b+c)^2}-\dfrac{b(c+a)}{(a-b+c)^2}+\dfrac{c(a+b)}{(a+b-c)^2}}, \quad \ldots \quad (1.93)$$

The transformation T_6 is given by $T_3\,T_2$ or $T_2\,T_3$ and implies

$$u^2 \to \frac{a^2}{(-3a^4+b^4+c^4-2b^2c^2+2c^2a^2+2a^2b^2)^2},\ \text{etc,.}$$

The transformations T_5, T_7, T_{11}, etc. will be obtained by a different method in the subsequent work.

2

We have already established quadratic involutions connecting t_i, t_{i+1}; t_i, t_{i+2}; t_i, t_{i+3}; t_i, t_{i+4}. Let us suppose that further quadratic involutions have been established connecting t_i, t_{i+5}; t_i, t_{i+6}; . . . ; t_i, t_{i+r-1}. Now r may either be of the form $2s$ or of the form $2s+1$. If $r=2s$, the involution connecting t_i, t_{i+r} could be directly deduced by operating with T_2 on the involution connecting t_i, t_{i+s}. If $r=2s+1$, we may eliminate t_{i+s} between the two involutions connecting respectively t_i, t_{i+s} and t_{i+s}, t_{i+r} and reject the retrogressing factor:

$$u^2(t_i+t_{i+r})^2+v^2(1-t_i\,t_{i+r})^2-w^2(1+t_i\,t_{i+r})^2.$$

Hence is obtained the involution connecting t_i, t_{i+r}. The induction is thus completed and we have proved the existence (for all values of r) of a quadratic involution of the form

$$a_r^2(t_i+t_{i+r})^2+b_r^2(1-t_i\,t_{i+r})^2 = c_r^2(1+t_i\,t_{i+r})^2 \quad \ldots \quad (2.1)$$

It follows that

$$t_{i+r}\,.\,t_{i-r} = \frac{t_i^2-t_r^2}{1-t_i^2\,t_r^2} \quad \ldots \quad (2.2)$$

for all values of i and r, provided we agree that $t_{-r} = -t_r$ and that $u^2 \rightarrow a_r^2$, $v^2 \rightarrow b_r^2$, $w^2 \rightarrow c_r^2$ defines the general replicating transformation T_r.

3

From the initial assumption $\theta_0=0$, it follows that if an n-agon be inscribed in S and circumscribed about s, the n-agon will lie symmetrically about the line joining the centres of the two circles.

If $n=2m$, then θ_m would be an odd multiple of π and hence $t_m=\pm 1$. Since the polygon in this case is a genuine one, the medial diagonals must concur at a finite point and hence the porism given by $t_m^2=1$ can only be the porism corresponding to the "inseparable" case. Hence $K(2m)$ is simply given by $t_m^2=1$.

To obtain the porism of $(2m+1)$-agon, we need only remember that θ_m, θ_{m+1} must respectively be of the forms $(2r+1)\pi-\alpha$ and $(2r+1)\pi+\alpha$. It follows that the porism $C_2(2m+1)$ is given by $t_m\,t_{m+1}=1$. But from (2.2),

$$\frac{t_{m+1}^2-t_m^2}{1-t_{m+1}^2\,t_m^2} = t_1 t_{2m+1},$$

where the denominator vanishes but the numerator does not as $t_{m+1}^2 = t_m^2$ would imply α to be a multiple of 2π, which is out of question. As t_1 is finite, it follows that $t_{2m+1} = \infty$ must also lead to the same porism.

We could also deduce the porism from the general condition $t_{m-j}\,t_{m+j+1} = 1$. In each individual case, it will have to be decided which of these various conditions would lead to the porism most rapidly. But when $(2m+1)$ is prime, the simplest method is to set $t_m\,t_{m+1} = 1$. When $(2m+1)$ is composite, the problem must become troublesome, as irrelevant factors are bound to enter into the porism and have to be removed at the cost of great labour, before the relevant porism is ready for exhibition.

Another point in this connection may also be mentioned. The porism thus obtained relates to the in-circle. The porism $C_1(2m+1)$ corresponding to the ex-circle is rapidly deduced by changing ρ into $-\rho$ in $C_2(2m+1)$.

We next come to the porism of a $(4m+2)$-agon for the "separable" case. The relevant condition can be deduced by considering only one half of the polygon, which lies symmetrically with respect to the common diameter. The situation in view can only arise if $\theta_m=\theta_{m+1}$ and consequently the polygon has to be retraced to the starting-point, and then to be continued on the other side of the bounding line. This implies that $t_m=t_{m+1}$ is the condition for the porism of a "separable" case. This gives $C_2(4m+2)$. The condition $C_1(4m+2)$ may be deduced from $C_2(4m+2)$ by merely changing the sign of c. The transformation T_2 can rapidly build up all porisms of the type $C(2^\gamma(2m+1))$. The only case that would not be covered by the argument would be the case $C(2^\gamma)$, but T_2 supplies all the relevant porisms if only $C(4)$ be known. To obtain $C(4)$, it is only necessary to observe that A_1 is one of the common points of s and S, and this can only happen if $R=\pm d$. The condition $R-d=0$ will be known as $C_2(4)$ and the condition $R+d=0$ as $C_1(4)$, although in this particular case the change of sign of c to convert the one form into the other becomes meaningless.

4

The various formulæ connecting the t's have been used to obtain the values of t_i up to and including $i=16$. To condense the formulæ, the following abbreviations have been used:

$$e_0 \equiv a+b+c,$$
$$e_1 \equiv -a+b+c,$$
$$e_2 \equiv a-b+c,$$
$$e_3 \equiv a+b-c;$$
$$E_1 \equiv -a^2+b^2+c^2,$$
$$E_2 \equiv a^2-b^2+c^2,$$
$$E_3 \equiv a^2+b^2-c^2.$$
$$f_0 \equiv aE_1+bE_2+cE_3+2abc$$
$$f_1 \equiv -aE_1+bE_2+cE_3-2abc$$
$$f_2 \equiv aE_1-bE_2+cE_3-2abc$$
$$f_3 \equiv aE_1+bE_2-cE_3-2abc$$
$$F_1 \equiv -3a^4+b^4+c^4-2b^2c^2+2c^2a^2+2a^2b^2 \equiv -E_2E_3+E_3E_1+E_1E_2$$
$$F_2 \equiv a^4-3b^4+c^4+2b^2c^2-2c^2a^2+2a^2b^2 \equiv E_2E_3-E_3E_1+E_1E_2$$
$$F_3 \equiv a^4+b^4-3c^4+2b^2c^2+2c^2a^2-2a^2b^2 \equiv E_2E_3+E_3E_1-E_1E_2$$
$$E_0 \equiv -a^4-b^4-c^4+2b^2c^2+2c^2a^2+2a^2b^2 \equiv E_2E_3+E_3E_1+E_1E_2 \equiv e_0e_1e_2e_3$$
$$g_0 \equiv E_1E_2E_3+2abE_1E_2+2bcE_2E_3+2caE_3E_1$$
$$g_1 \equiv E_1E_2E_3-2abE_1E_2+2bcE_2E_3-2caE_3E_1$$
$$g_2 \equiv E_1E_2E_3-2abE_1E_2-2bcE_2E_3+2caE_3E_1$$
$$g_3 \equiv E_1E_2E_3+2abE_1E_2-2bcE_2E_3-2caE_3E_1$$

$$G_1 \equiv -E_2{}^2E_3{}^2+E_3{}^2E_1{}^2+E_1{}^2E_2{}^2 \equiv (-a^2+b^2+c^2)^4-16b^2c^2(b^2-a^2)(c^2-a^2)$$
$$G_2 \equiv \quad E_2{}^2E_3{}^2-E_3{}^2E_1{}^2+E_1{}^2E_2{}^2 \equiv (a^2-b^2+c^2)^4-16c^2a^2(c^2-b^2)(a^2-b^2)$$
$$G_3 \equiv \quad E_2{}^2E_3{}^2+E_3{}^2E_1{}^2-E_1{}^2E_2{}^2 \equiv (a^2+b^2-c^2)^4-16a^2b^2(a^2-c^2)(b^2-c^2)$$
$$h_0 \equiv \quad aF_2F_3+bF_3F_1+cF_1F_2$$
$$h_1 \equiv -aF_2F_3+bF_3F_1+cF_1F_2$$
$$h_2 \equiv \quad aF_2F_3-bF_3F_1+cF_1F_2$$
$$h_3 \equiv \quad aF_2F_3+bF_3F_1-cF_1F_2$$
$$H_1 \equiv -G_1E_2E_3+G_2E_3E_1+G_3E_1E_2-2E_1{}^2E_2{}^2E_3{}^2 \equiv \tfrac{1}{2}[F_1F_2F_3-F_0(-F_2F_3+F_3F_1+F_1F_2)]$$
$$H_2 \equiv \quad G_1E_2E_3-G_2E_3E_1+G_3E_1E_2-2E_1{}^2E_2{}^2E_3{}^2 \equiv \tfrac{1}{2}[F_1F_2F_3-F_0(F_2F_3-F_3F_1+F_1F_2)]$$
$$H_3 \equiv \quad G_1E_2E_3+G_2E_3E_1-G_3E_1E_2-2E_1{}^2E_2{}^2E_3{}^2 \equiv \tfrac{1}{2}[F_1F_2F_3-F_0(F_2F_3+F_3F_1-F_1F_2)]$$
$$H_0 \equiv \quad G_1E_2E_3+G_2E_3E_1+G_3E_1E_2+2E_1{}^2E_2{}^2E_3{}^2 \equiv +f_0f_1f_2f_3$$
$$i_0 \equiv \quad F_0(aE_1F_2F_3+bE_2F_3F_1+cE_3F_1F_2)+2abcF_1F_2F_3$$
$$i_1 \equiv \quad F_0(-aE_1F_2F_3+bE_2F_3F_1+cE_3F_1F_2)-2abcF_1F_2F_3$$
$$i_2 \equiv \quad F_0(aE_1F_2F_3-bE_2F_3F_1+cE_3F_1F_2)-2abcF_1F_2F_3$$
$$i_3 \equiv \quad F_0(aE_1F_2F_3+bE_2F_3F_1-cE_3F_1F_2)-2abcF_1F_2F_3$$
$$I_1 \equiv \quad T_2F_1 \equiv -G_2G_3+G_3G_1+G_1G_2$$
$$I_2 \equiv \quad T_2F_2 \equiv \quad G_2G_3-G_3G_1+G_1G_2$$
$$I_3 \equiv \quad T_2F_3 \equiv \quad G_2G_3+G_3G_1-G_1G_2$$
$$I_0 \equiv \quad T_2F_0 \equiv \quad F_0F_1F_2F_3$$
$$j_0 \equiv \quad F_0(aG_1F_2F_3+bG_2F_3F_1+cG_3F_1F_2)+4abcE_1E_2E_3F_1F_2F_3$$
$$j_1 \equiv \quad F_0(-aG_1F_2F_3+bG_2F_3F_1+cG_3F_1F_2)-4abcE_1E_2E_3F_1F_2F_3$$
$$j_2 \equiv \quad F_0(aG_1F_2F_3-bG_2F_3F_1+cG_3F_1F_2)-4abcE_1E_2E_3F_1F_2F_3$$
$$j_3 \equiv \quad F_0(aG_1F_2F_3+bG_2F_3F_1-cG_3F_1F_2)-4abcE_1E_2E_3F_1F_2F_3$$
$$J_1 \equiv -2E_1E_2E_3(-E_1G_2G_3+E_2G_3G_1+E_3G_1G_2)+G_1G_2G_3$$
$$J_2 \equiv -2E_1E_2E_3(E_1G_2G_3-E_2G_3G_1+E_3G_1G_2)+G_1G_2G_3$$
$$J_3 \equiv -2E_1E_2E_3(E_1G_2G_3+E_2G_3G_1-E_3G_1G_2)+G_1G_2G_3$$
$$J_0 \equiv \quad 2E_1E_2E_3(E_1G_2G_3+E_2G_3G_1+E_3G_1G_2)+G_1G_2G_3 \equiv -g_0g_1g_2g_3$$
$$k_0 \equiv \quad F_1F_2F_3e_1e_2e_3f_1f_2f_3-2E_1E_2E_3g_1g_2g_3$$
$$k_1 \equiv \quad F_1F_2F_3e_0e_2e_3f_0f_2f_3-2E_1E_2E_3g_0g_2g_3$$
$$k_2 \equiv \quad F_1F_2F_3e_0e_1e_3f_0f_3f_1-2E_1E_2E_3g_0g_3g_1$$
$$k_3 \equiv \quad F_1F_2F_3e_0e_1e_2f_0f_1f_2-2E_1E_2E_3g_0g_1g_2$$
$$K_1 \equiv -G_2{}^2G_3{}^2+G_3{}^2G_1{}^2+G_1{}^2G_2{}^2$$
$$K_2 \equiv \quad G_2{}^2G_3{}^2-G_3{}^2G_1{}^2+G_1{}^2G_2{}^2$$
$$K_3 \equiv \quad G_2{}^2G_3{}^2+G_3{}^2G_1{}^2-G_1{}^2G_2{}^2.$$

The t's may now be written as

$$t_0 = 0 \qquad t_1 = \left(\frac{c-b}{c+b}\right)^{\frac{1}{2}}$$

$$t_2 = \frac{c+b}{a}.t_1 \qquad t_3 = \frac{e_0e_1}{e_2e_3}\,t_1$$

$$t_4 = \frac{2a^2E_1}{E_2E_3}\,t_2 \qquad t_5 = -\frac{f_0f_1}{f_2f_3}\,t_1$$

$$t_6 = \frac{F_0F_1}{F_2F_3}\,t_2 \qquad t_7 = -\frac{g_0g_1}{g_2g_3}\,t_1$$

$$t_8 = \frac{4a^2E_1E_2E_3G_1}{G_2G_3}.t_2 \qquad t_9 = \frac{e_0e_1h_0h_1}{e_2e_3h_2h_3}\,t_1$$

$$t_{10} = -\frac{H_0H_1}{H_2H_3}\,t_2 \qquad t_{11} = \frac{i_0i_1}{i_2i_3}\,t_1$$

$$t_{12} = \frac{I_0I_1}{I_2I_3}\,t_4 = \frac{2a^2e_0e_1e_2e_3E_1F_1F_2F_3I_1}{E_2E_3I_2I_3}.t_2$$

$$t_{13} = -\frac{j_0j_1}{j_2j_3}\,t_1 \qquad t_{14} = -\frac{J_0J_1}{J_2J_3}\,t_2$$

$$t_{15} = -\frac{e_0e_1f_0f_1k_0k_1}{e_2e_3f_2f_3k_2k_3}\,t_1 \qquad t_{16} = \frac{8a^2E_1E_2E_3G_1G_2G_3K_1}{K_2K_3}\,t_2$$

The porisms obtainable by methods sketched above must all occur in their rationalized forms. To obtain Chaundy's irrational forms, we must apply the inverse transformation of T_0 and denote it by T_0^{-1}. This obviously implies

$$a \to u, \quad b \to v, \quad c \to w$$

It would accordingly be worth while to work out in terms of R, r, d the results operating with T_0^{-1} on some of the functions defined above. Once these have been calculated, we could almost instantaneously pass on from rational to irrational forms or vice versa. Remembering that

$$2u^2 = R-d+r, \quad 2v^2 = R+d+r, \quad 2w^2 = 2R,$$

it follows that:

$$T_0^{-1}E_1 = -u^2+v^2+w^2 = R+d = \frac{1}{a}$$

$$T_0^{-1}E_2 = u^2-v^2+w^2 = R-d = \frac{1}{b}$$

$$T_0^{-1}E_3 = u^2+v^2-w^2 = r = \frac{1}{c}$$

$$T_0^{-1}F_1 = R^2-d^2+2rd = \frac{e_1}{abc}$$

$$T_0^{-1}F_2 = R^2-d^2-2rd = \frac{e_2}{abc}$$

$$T_0^{-1}F_3 = d^2-R^2+2Rr = \frac{e_3}{abc}$$

$$T_0^{-1}F_0 = R^2-d^2+2Rr = \frac{e_0}{abc}$$

$$T_0^{-1}G_1 = (R^2-d^2)^2+4r^2Rd = E_1(abc)^{-2}$$

$$T_0^{-1}G_2 = (R^2-d^2)^2-4r^2Rd = E_2(abc)^{-2}$$

$$T_0^{-1}G_3 = 2r^2(R^2+d^2)-(R^2-d^2)^2 = E_3(abc)^{-2}$$

$$T_0^{-1}H_1 = 2rd(R^2-d^2)(R^2-d^2+2rd)-(R^2-d^2)^3-8r^3R^2d = f_1(abc)^{-3}$$

$$T_0^{-1}H_2 = 2rd(R^2-d^2)(d^2-R^2+2rd)-(R^2-d^2)^3+8r^3R^2d = f_2(abc)^{-3}$$

$$T_0^{-1}H_3 = 2Rr(R^2-d^2)(R^2-d^2-2Rr)+(R^2-d^2)^3-8r^2Rd^2 = f_3(abc)^{-3}$$

$$T_0^{-1}H_0 = 2Rr(R^2-d^2)(R^2-d^2+2Rr)-(R^2-d^2)^3-8r^3Rd^2 = f_0(abc)^{-3}$$

$$T_0^{-1}I_1 = F_1(abc)^{-4}$$

$$T_0^{-1}I_2 = F_2(abc)^{-4}$$

$$T_0^{-1}I_3 = F_3(abc)^{-4}$$

$$T_0^{-1}I_0 = F_0(abc)^{-4}$$

$$T_0^{-1}J_1 = g_1(abc)^{-6}$$

$$T_0^{-1}J_2 = g_2(abc)^{-6}$$

$$T_0^{-1}J_3 = g_3(abc)^{-6}$$

$$T_0^{-1}J_0 = g_0(abc)^{-6}$$

$$T_0^{-1}K_1 = G_1(abc)^{-8}$$

$$T_0^{-1}K_2 = G_2(abc)^{-8}$$

$$\begin{aligned} T_0^{-1}K_3 &= G_3(abc)^{-8} \\ &= \{(R^2-d^2)^2-2r^2(R^2+d^2)\}^4-16r^4(R^2-d^2)^4\{(R^2+r^2-d^2)^2-4R^2r^2\}. \end{aligned}$$

5

We are now fully equipped to proceed to the calculation of a number of porisms. $C_2(3)$ is given by

$$t_1t_2 = 1 \quad \text{or} \quad a+b = c \quad \text{or} \quad (R+d)^{-1}+(R-d)^{-1} = r^{-1}$$

On applying T_0^{-1}, we get Chaundy's irrational form

$$\sqrt{(R-d+r)}+\sqrt{(R+d+r)} = \sqrt{(2R)}$$

On changing the sign of c, we obtain $C_1(8)$ in the form

$$a+b+c=0 \quad \text{or} \quad e_0=0 \quad \text{or} \quad (R+d)^{-1}+(R-d)^{-1}+r^{-1}=0.$$

$C_2(6)$ is given by

$$t_1=t_2 \quad \text{or} \quad -a+b+c=0 \quad \text{or} \quad e_1=0$$

and, therefore, $C_1(6)$ by

$$a-b+c=0 \quad \text{or} \quad e_2=0.$$

We have already seen that $C_2(4)$ is given by $R=d$ and $C_1(4)$ by $R+d=0$. $K(4)$ is given by $t_4^2=1$ or $a^2+b^2=c^2$ or $E_3=0$, $C_1(8)$ is given by T_2 on $R+d=0$ leading to $E_2=0$ and hence $C_2(8) \equiv E_1=0$

$C_2(5)$ is given by $t_2t_3=1$ which leads to $f_3=0$

Hence

$$C_1(5) \equiv f_0=0, \quad C_1(10) \equiv f_2=0, \quad C_2(10) \equiv f_1=0$$

Also

$$T_0^{-1}f_0 \equiv u(-u^2+v^2+w^2)+v(u^2-v^2+w^2)+w(u^2+v^2-w^2)+2uvw=0$$

which is Chaundy's irrational form for $C_1(5)$, *viz.*,

$$\frac{u}{a}+\frac{v}{b}+\frac{w}{c}+2uvw=0$$

The various ways of writing $f_0=0$ are

$$(a+b)(b+c)(c+a)=a^3+b^3+c^3,$$

$$(a+b+c)^3=4(a^3+b^3+c^3),$$

$$(-a+b+c)(a-b+c)(a+b-c)+4abc=0,$$

$$\begin{vmatrix} e_0 & e_3 & e_2 \\ e_3 & e_0 & e_1 \\ e_2 & e_1 & e_0 \end{vmatrix}=0,$$

$$a(-a^2+b^2+c^2)+b(a^2-b^2+c^2)+c(a^2+b^2-c^2)+2abc=0,$$

$$e_0^{-1}+e_1^{-1}+e_2^{-1}+e_3^{-1}=0, \quad \text{etc.}$$

$K(6)$ is given by T_2 on $C_2(3)$ or T_2 on e_3 or $F_3=0$. T_2 on $C_1(3)$ gives $e_0e_1e_2e_3$ which gives nothing new. $K(6)$ may also be written

$$E_1^{-1}+E_2^{-1}=E_3^{-1}$$

$C_1(12)$ is given by T_2 on $C_1(6)$ or $F_2=0$. $C_2(12)$ is accordingly $F_1=0$. $C_2(7)$ is given by $t_1t_4=1$ giving $g_3=0$. Hence

$$C_1(7) \equiv g_0=0, \quad C_1(14) \equiv g_2=0 \quad \text{and} \quad C_2(14) \equiv g_1=0.$$

$T_0^{-1}g_0=0$ gives

$$\frac{a}{u}+\frac{b}{v}+\frac{c}{w}+\frac{1}{2uvw}=0$$

which is Chaundy's irrational representative and which excites admiration when compared with the corresponding representative for a pentagon. It follows that the porism for a heptagon can be exhibited in each of the six rational forms corresponding to the six forms given above for a pentagon.

$K(8)$ is given by T_2 on $K(4)$ or $G_3 = 0$ or $E_1^{-2}+E_2^{-2} = E_3^{-2}$

and

$C_1(16)$ is given by T_2 on $C_1(8)$ or $G_2 = 0$ or $E_1^{-2}+E_3^{-2} = E_2^{-2}$

$C_2(16)$ is given by T_2 on $C_2(8)$ or $G_1 = 0$ or $E_2^{-2}+E_3^{-2} = E_1^{-2}$

$C_1(9)$ may be derived from T_3 on $C_1(3)$ or $h_0 = 0$.

Hence

$$C_2(9) \equiv h_3 = 0, \quad C_1(18) \equiv h_2 = 0, \quad \text{and} \quad C_2(18) \equiv h_1 = 0$$

h_0 may also be exhibited in the form

$$\frac{a}{F_1}+\frac{b}{F_2}+\frac{c}{F_3} = 0,$$

which is changed by T_0^{-1} into Chaundy's irrational representative, *viz.*,

$$\frac{\sqrt{(R-d+r)}}{R^2-d^2+2rd}+\frac{\sqrt{(R+d+r)}}{R^2-d^2-2rd}+\frac{\sqrt{(2R)}}{d^2-R^2+2Rr} = 0$$

$K(10)$ may best be obtained by T_2 on $f_3 = 0$ which turns into

$$H_3 = 0 \quad \text{or} \quad \frac{G_1}{E_1}+\frac{G_2}{E_2}+\frac{G_3}{E_3}+2E_1E_2E_3 = 0.$$

Accordingly,

$$C_1(20) \equiv H_2 = 0, \qquad C_2(20) \equiv H_1 = 0.$$

T_2 on $f_0=0$ gives $H_0=0$ or $f_0f_1f_2f_3=0$ which is nothing new.

We come next to the undecagon, which is given by $t_5t_6=1$ or $f_0f_1e_1e_2e_3F_1(b-c) = af_2f_3F_2F_3$. This may be rewritten in the form $i_3=0$ or

$$\frac{aE_1}{F_1}+\frac{bE_2}{F_2}-\frac{cE_3}{F_3}=\frac{2abc}{F_0}$$

Hence $C_1(11)\equiv i_0=0$ or

$$\frac{aE_1}{F_1}+\frac{bE_2}{F_2}+\frac{cE_3}{F_3}+\frac{2abc}{F_0} = 0.$$

On applying T_0^{-1} we obtain the irrational representative

$$\frac{u}{ae_1}+\frac{v}{be_2}+\frac{w}{ce_3}+\frac{2uvw}{e_0} = 0,$$

or more explicitly

$$\frac{(R+d)\sqrt{(R+r-d)}}{R^2-d^2+2rd}+\frac{(R-d)\sqrt{(R+r+d)}}{R^2-d^2-2rd}+\frac{r\sqrt{(2R)}}{d^2-R^2+2Rr}+\frac{\sqrt{\{2R(R+r+d)(R+r-d)\}}}{R^2-d^2+2Rr} = 0.$$

This may be compared with Chaundy's result. It is obvious that Chaundy's result contains two errors. In the denominators of the first two terms of the porism as exhibited above; Chaundy's porism contains $+2Rd$ and $-2Rd$, whereas these ought to be $+2rd$ and $-2rd$ respectively. With this modification, Chaundy's result tallies completely with the one given here. That these represent actual errors in calculation and not merely misprints follows from an inspection of $C(22)$, $K(22)$ and $C(44)$, all of which are tainted by the same two errors in $C(11)$. Consequently, Chaundy's $K(44)$—which he considers unprintable—must contain the same errors, as also every other porism deduced by him from $C(11)$.

It follows that $C_1(22)\equiv i_2=0$ and $C_2(22)\equiv i_1=0$. I exhibit $C_1(22)$ in its irrational form so that comparison may be made with Chaundy's result

$$\frac{(R+d)\sqrt{(d+r-R)}}{d^2-R^2+2rR}+\frac{(d-R)\sqrt{(d+r+R)}}{d^2-R^2-2rR}+\frac{r\sqrt{(2d)}}{R^2-d^2+2rd}+\frac{\sqrt{\{2d(d+r+R)(d+r-R)\}}}{d^2-R^2+2dr}=0$$

The correct forms of $C(11)$ and $C(22)$, both rational and irrational, appear, I believe, for the first time in this paper.

The majority of the porisms which follow are believed to have been given here for the first time in their explicit forms. We proceed further. $K(12)$ would result from T_2 on $K(6)$ or

$$T_2F_3=0 \quad \text{or} \quad I_3=0 \quad \text{or} \quad G_1^{-1}+G_2^{-1}=G_3^{-1}.$$

Hence

$$C_1(24)\equiv I_2=0 \quad \text{and} \quad C_2(24)\equiv I_1=0.$$

The porism of a 13-agon is furnished by $t_6t_7=1$ leading to

$$(b-c)F_0F_1g_0g_1=ag_2g_3F_2F_3 \quad \text{or} \quad j_3=0,$$

Hence

$$C_1(13)\equiv j_0=0, \quad C_1(26)\equiv j_2=0 \quad \text{and} \quad C_2(26)\equiv j_1=0$$

$j_0=0$ may also be written as

$$\frac{aG_1}{F_1}+\frac{bG_2}{F_2}+\frac{cG_3}{F_3}+\frac{4abcE_1E_2E_3}{F_0}=0.$$

On applying the irrationalizing transformation, this becomes

$$\frac{uE_1}{e_1}+\frac{vE_2}{e_2}+\frac{wE_3}{e_3}+\frac{4uvwabc}{e_0}=0$$

or more explicitly

$$\frac{\{(R^2-d^2)^2+4r^2Rd\}\sqrt{(R-d+r)}}{R^2-d^2+2rd}+\frac{\{(R^2-d^2)^2-4r^2Rd\}\sqrt{(R+d+r)}}{R^2-d^2-2rd}$$

$$+\frac{\{2r^2(R^2+d^2)-(R^2-d^2)^2\}\sqrt{(2R)}}{d^2-R^2+2Rr}+\frac{\sqrt{\{2R(R-d+r)(R+d+r)\}}2r(R^2-d^2)}{R^2-d^2+2Rr}=0.$$

We next attack the 14-agon. $K(14)$ will be given by either $t_7^2=1$ or by T_3 on $C_1(7)$ or by $T_2g_3=0$ or $J_3=0$. Accordingly

$$C_1(28)\equiv J_2=0 \quad \text{and} \quad C_2(28)\equiv J_1=0$$

$K(14)$ may be exhibited in the form

$$\frac{E_1}{G_1}+\frac{E_2}{G_2}-\frac{E_3}{G_3}=\frac{1}{2E_1E_2E_3}.$$

The hurdle provided by a 15-agon is more troublesome. Chaundy just clears it, though not too gracefully, in the second attempt. The source of trouble springs from the two degenerate relatives, the triangle and the pentagon, who manage to creep in singly or together; no matter how we attack the problem. If we set $t_8t_7=1$, the triangle and the pentagon both creep in along with the 15-agon; if we set $t_5t_{10}=1$, the triangle alone is present and if we set $t_9t_6=1$, the pentagon keeps company with the 15-agon. If we apply T_3 on C_5, the pentagon is present and even if we knew the transformation T_5, T_5 on C_3 would bring in the undesirable triangle. One might have been tempted to give up the 15-agon as hopeless and to run away from the heavy algebra which blocks the way. But Chaundy's result gives rise to a hope that the algebra would simplify tremendously, if gracefully performed, and so we gather courage and proceed.

We set $t_5t_{10}=1$, which may be considered to result out of T_5 on $t_1t_2=1$. The result simplifies to

$$aH_2H_3 = (b-c)f_0^2f_1^2H_1,$$

which may be rewritten as

$$aH_2H_3+bH_3H_1-cH_1H_2 = 0$$

The symmetric porism must, therefore, run

$$aH_2H_3+bH_3H_1+cH_1H_2 = 0,$$

a symmetric expression of degree 25 in a, b, c; the redundant factor $a+b+c$ still remains to be weeded out. Before we plunge into the algebra, we note that

$$\frac{a}{H_1}+\frac{b}{H_2}+\frac{c}{H_3} = 0,$$

could also have resulted out of T_5 on $a+b+c=0$, and accordingly T_5 must be given by

$$a \to \frac{a}{H_1}, \quad b \to \frac{b}{H_2}, \quad c \to \frac{c}{H_3}.$$

We have here stumbled across a gem, which we were not looking for. Besides providing us with a tool far more powerful than T_3, it gives us a clue to the general law of replicating p times when p is a prime exceeding 2. We have only to note that T_3 implies $c \to c/K(6)$ and that T_5 involves $c \to c/K(10)$ to conjecture that in general T_p should involve $c > c/K(2p)$. We shall have occasion later on to test this conjecture for $p=7$, when we come to $C(21)$.

On trying to remove the factor $a+b+c$, we find

$$aH_2H_3+bH_3H_1+cH_1H_2 \equiv (a+b+c)(e_1f_1F_1e_2f_2F_2e_3f_3F_3-2E_1g_1E_2g_2E_3g_3)$$

Hence $C_1(15)$ is given by $k_0=0$. Also $C_2(15) \equiv k_3=0$, $C_1(30) \equiv k_2 = 0$, $C_2(30) \equiv k_1 = 0$. Now

$$\begin{aligned} k_0 &\equiv e_1e_2e_3f_1f_2f_3F_1F_2F_3-2g_1g_2g_3E_1E_2E_3 \\ &\equiv G_1G_2G_3-2\sum ab(F_1F_2F_3G_3\,\overline{E_1+E_2}+H_2E_3\,\overline{G_1+G_2}) \\ &\equiv G_1G_2G_3-4\sum ab(c^2F_1F_2F_3G_3+H_3E_3E_1^2E_2^2). \end{aligned}$$

Hence $C_1(15)$ is given by

$$G_1G_2G_3 = 4abcF_1F_2F_3(aG_1+bG_2+cG_3)+4E_1E_2E_3(abE_1E_2H_3+bcE_2E_3H_1+caE_3E_1H_2)$$

Operating with T_0^{-1} leads to the irrational representative

$$\frac{E_1E_2E_3}{4uvwabc} = e_1e_2e_3(uE_1+vE_2+wE_3)+\frac{af_1}{u}+\frac{bf_2}{v}+\frac{cf_3}{w}.$$

$$K(16) \equiv T_2K(8) \equiv T_2G_3 \equiv K_3 = 0$$

or

$$G_1^{-2}+G_2^{-2} = G_3^{-2}.$$

Hence

$$C_1(32) \equiv K_2 = 0 \quad \text{and} \quad C_2(32) \equiv K_1 = 0.$$

The 17-agon yields the porism on setting $t_0t_8=1$. This gives

$$C_2(17) \equiv 4a(c-b)e_0e_1h_0h_1E_1E_2E_3G_1-e_2e_3h_2h_3G_2G_3 = 0.$$

Hence

$$C_1(17) \equiv 4a(c+b)e_2e_3h_2h_3E_1E_2E_3G_1+e_0e_1h_0h_1G_2G_3 = 0,$$

$$C_1(34) \equiv 4a(c-b)e_0e_1h_0h_1E_1E_2E_3G_1+e_2e_3h_2h_3G_2G_3 = 0,$$

and

$$C_2(34) \equiv 4a(c+b)e_2e_3h_2h_3E_1E_2E_3G_1-e_0e_1h_0h_1G_2G_3 = 0.$$

T_2 on $C_2(9)$ gives $K(18)$. Hence

$$K(18) \equiv (E_1I_1)^{-1}+(E_2I_2)^{-1}-(E_3I_3)^{-1} = 0,$$

$$C_1(36) \equiv (E_1I_1)^{-1}-(E_2I_2)^{-1}+(E_3I_3)^{-1} = 0,$$

and

$$C_2(36) \equiv -(E_1I_1)^{-1}+(E_2I_2)^{-1}+(E_3I_3)^{-1} = 0.$$

The next polygon is the 19-agon. This is given by $t_0t_{10} = 1$ leading to

$$C_2(19) \equiv (b-c)e_0e_1h_0h_1F_0H_1+ae_2e_3h_2h_3H_2H_3 = 0,$$

Hence

$$C_1(19) \equiv (b+c)e_2e_3h_2h_3F_0H_1+ae_0e_1h_0h_1H_2H_3 = 0,$$

$$C_2(38) \equiv (b+c)e_2e_3h_2h_3F_0H_1-ae_0e_1h_0h_1H_2H_3 = 0,$$

and

$$C_1(38) \equiv (b-c)e_0e_1h_0h_1F_0H_1-ae_2e_3h_2h_3H_2H_3 = 0.$$

T_2 on $K(10)$ implies T_2 on $H_3 = 0$. Hence

$$K(20) \equiv \frac{K_1}{G_1}+\frac{K_2}{G_2}-\frac{K_3}{G_3}-2G_1G_2G_3 = 0,$$

$$C_1(40) \equiv \frac{K_1}{G_1}-\frac{K_2}{G_2}+\frac{K_3}{G_3}-2G_1G_2G_3 = 0,$$

$$C_2(40) \equiv -\frac{K_1}{G_1}+\frac{K_2}{G_2}+\frac{K_3}{G_3}-2G_1G_2G_3 = 0.$$

The 21-agon proves to be as troublesome as the 15-agon, and has to be attacked by similar methods.

$$T_7 \text{ on } t_2t_1 = 1 \text{ gives } t_{14}t_7 = 1,$$

which simplifies to

$$aJ_2J_3 + (c-b)g_0{}^2g_1{}^2J_1 = 0$$

or

$$\frac{a}{J_1} + \frac{b}{J_2} - \frac{c}{J_3} = 0$$

This could have been deduced by applying T_7 on $a+b-c = 0$ for a triangle. Hence T_7 is defined by

$$a \rightarrow \frac{a}{J_1}, \quad b \rightarrow \frac{b}{J_2}, \quad c \rightarrow \frac{c}{J_3},$$

which corroborates our conjecture regarding the general law of transformation for replicating a prime number of times.

As in the case of a 15-agon, the porism is tainted by the presence of an irrelevant factor $a+b-c$ corresponding to a triangle. Now

$$aJ_2J_3 + bJ_3J_1 + cJ_1J_2 \equiv (a+b+c)[-4E_1E_2E_3G_1G_2G_3f_1f_2f_3h_0 + F_1F_2F_3e_1e_2e_3g_1g_2g_3i_0]$$

Hence $C_1(21)$ is given by

$$4E_1E_2E_3G_1G_2G_3f_1f_2f_3h_0 - e_1e_2e_3E_1F_2F_3g_1g_2g_3i_0 = 0$$

$$\therefore \quad C_2(21) \equiv 4E_1E_2E_3G_1G_2G_3f_0f_1f_2h_3 - e_0e_1e_2F_1F_2F_3g_0g_1g_2i_3 = 0;$$

$$C_1(42) \equiv 4E_1E_2E_3G_1G_2G_3f_0f_1f_3h_2 - e_0e_1e_3F_1F_2F_3g_0g_1g_3i_2 = 0,$$

and

$$C_2(42) \equiv 4E_1E_2E_3G_1G_2G_3f_0f_2f_3h_1 - e_0e_2e_3F_1F_2F_3g_0g_2g_3i_1 = 0.$$

The next porism concerns a 22-agon. T_2 on $i_3 = 0$ gives $K(22)$ in the form

$$\frac{G_1}{E_1I_1} + \frac{G_2}{E_2I_2} - \frac{G_3}{E_3I_3} - \frac{2E_1E_2E_3}{F_0F_1F_2F_3} = 0$$

Hence

$$C_1(44) \equiv \frac{G_1}{E_1I_1} - \frac{G_2}{E_2I_2} + \frac{G_3}{E_3I_3} - \frac{2E_1E_2E_3}{F_0F_1F_2F_3} = 0$$

and

$$C_2(44) \equiv -\frac{G_1}{E_1I_1} + \frac{G_2}{E_2I_2} + \frac{G_3}{E_3I_3} - \frac{2E_1E_2E_3}{F_0F_1F_2F_3} = 0.$$

On setting $t_{11}t_{12} = 1$, we obtain $C_2(23)$ in the form

$$2a(c-b)E_1F_0F_1F_2F_3I_1i_0i_1 - E_2E_3I_2I_3i_2i_3 = 0$$

from which

$$C_1(23) \equiv 2a(b+c)F_0F_1F_2F_3E_1I_1i_2i_3 + E_2E_3I_2I_3i_0i_1 = 0,$$

$$C_1(46) \equiv 2a(b-c)F_0F_1F_2F_3E_1I_1i_0i_1 - E_2E_3I_2I_3i_2i_3 = 0,$$

and

$$C_2(46) \equiv -2a(b+c)F_0F_1F_2F_3E_1I_1i_2i_3 + E_2E_3I_2I_3i_0i_1 = 0.$$

$K(24) \equiv T_2K(12) \equiv I_2I_3 = 0$ giving

$$K_1{}^{-1} + K_2{}^{\ 1} - K_3{}^{-1} = 0$$

Hence

$$C_1(48) \equiv K_1^{-1} - K_2^{-1} + K_3^{-1} = 0$$

and

$$C_2(48) \equiv -K_1^{-1} + K_2^{-1} + K_3^{-1} = 0$$

$C_1(25)$ is obtained by T_5 on $C_1(5)$ leading immediately to

$$C_1(25) \equiv (e_0k_0)^{-1} + (e_1k_1)^{-1} + (e_2k_2)^{-1} + (e_3k_3)^{-1} = 0.$$

$$C_2(25) \equiv (e_0k_0)^{-1} - (e_1k_1)^{-1} - (e_2k_2)^{-1} + (e_3k_3)^{-1} = 0,$$

$$C_1(50) \equiv (e_0k_0)^{-1} - (e_1k_1)^{-1} + (e_2k_2)^{-1} - (e_3k_3)^{-1} = 0,$$

and

$$C_2(50) \equiv (e_0k_0)^{-1} + (e_1k_1)^{-1} - (e_2k_2)^{-1} - (e_3k_3)^{-1} = 0.$$

We proceed no further; not because further progress has become more difficult, but because the porisms cannot present any new features of interest. The same forms are observed to repeat themselves. To pursue the matter further is merely a matter of algebra and patience.

To demonstrate the power of the tools in our hands, we work out $C_1(121)$. This will be given by T_{11} on $i_0 = 0$. To define T_{11}, we set

$$L_1 \equiv F_0F_1F_2F_3(-G_1E_2E_3I_2I_3 + G_2E_3E_1I_3I_1 + G_3E_1E_2I_1I_2) - 2E_1^2E_2^2E_3^2$$

$$L_2 \equiv F_0F_1F_2F_3(G_1E_2E_3I_2I_3 - G_2E_3E_1I_3I_1 + G_3E_1E_2I_1I_2) - 2E_1^2E_2^2E_3^2$$

and

$$L_3 \equiv F_0F_1F_2F_3(G_1E_2E_3I_2I_3 + G_2E_3E_1I_3I_1 - G_3E_1E_2I_1I_2) - 2E_1^2E_2^2E_3^2$$

Then $C_1(121)$ is given by

$$\sum \frac{a\left[-\dfrac{a^2}{L_1^2} + \dfrac{b^2}{L_2^2} + \dfrac{c^2}{L_3^2}\right]}{L_1\left[-\dfrac{3a^4}{L_1^4} + \dfrac{b^4}{L_2^4} + \dfrac{c^4}{L_3^4} - \dfrac{2b^2c^2}{L_2^2L_3^2} + \dfrac{2c^2a^2}{L_3^2L_1^2} + \dfrac{2a^2b^2}{L_1^2L_2^2}\right]}$$

$$+ \frac{2abc}{\left(\dfrac{a}{L_1} + \dfrac{b}{L_2} + \dfrac{c}{L_3}\right)\left(-\dfrac{a}{L_1} + \dfrac{b}{L_2} + \dfrac{c}{L_3}\right)\left(\dfrac{a}{L_1} - \dfrac{b}{L_2} + \dfrac{c}{L_3}\right)\left(\dfrac{a}{L_1} + \dfrac{b}{L_2} - \dfrac{c}{L_3}\right)} = 0$$

This is homogeneous and symmetric in a, b, c and of degree 1815. Its irrational representative can be worked out without much trouble, if desired. $C_2(121)$, $C_1(242)$ and $C_2(242)$ follow readily on proper sign adjustments.

If our ambition had been a little more unbridled, we would have aimed at $C_1(1681)$. The method would have been to calculate t_{20} by T_2 on t_{10} and t_{21} by T_3 on t_7. Thus t_{21} t_{20} equated to 1 would yield $C_2(41)$, from which $C_1(41)$ could immediately be written down. Then T_2 on $C_2(41)$ would yield $K(82)$, from which $C_1(164)$, $C_2(164)$ would come out by proper sign-adjustments. Then T_{41} is defined by

$$a \rightarrow \frac{a}{C_2(184)}, \quad b \rightarrow \frac{b}{C_1(184)}, \quad c \rightarrow \frac{c}{K(82)}$$

T_{41} on $C_1(41)$ yields $C_1(1681)$. But an actual working out would not be very profitable, although no insuperable difficulties crop up in the algebra. The final expression would be homogeneous and symmetrical in a, b, c and its degree would be 353,010. By such methods $K(2 \cdot 23^2)$ or $K(1058)$ and many similar porisms come out with great ease of calculation.

6

Considerations of completeness impel us to introduce elliptic functions at this conclusive stage. The treatment is in principle suggested by Clifford's work on two-two correspondences. Let

$$f(t,\, t') \equiv a(b-c)(1+t^2t'^2)-4bctt'+a(b+c)(t^2+t'^2) = 0,$$

$$\therefore \quad dt \Big/ \frac{\partial f}{\partial t'} + dt' \Big/ \frac{\partial f}{\partial t} = 0$$

or

$$\frac{dt}{\sqrt{(1+2\lambda t^2+t^4)}} = \frac{dt'}{\sqrt{(1+2\lambda t'^2+t'^4)}}$$

where

$$\lambda = 1+\frac{2c^2(a^2-b^2)}{a^2(b^2-c^2)}.$$

We assume $\lambda > 1$ and set

$$\lambda = \cosh\omega, \qquad x = \frac{te^{\omega/2}}{\sqrt{(1+t^2e^{\omega})}} \qquad \text{and} \qquad x' = \frac{t'e^{\omega/2}}{\sqrt{(1+t'^2e^{\omega})}}$$

This leads to

$$\int \frac{dx}{\sqrt{\{(1-x^2)(1-x^2k^2)\}}} = \int \frac{dx'}{\sqrt{\{(1-x^2)(1-k^2x'^2)\}}}$$

where $k^2 = 1-e^{-2\omega}$, yielding

$$\operatorname{sn}^{-1}(x,\, k)-\operatorname{sn}^{-1}(x',\, k) = \text{constant}$$

Suppose $x' = 0$ gives $x = x_1$. Hence

$$\operatorname{sn}^{-1}(x,\, k)-\operatorname{sn}^{-1}(x',\, k) = \operatorname{sn}^{-1}(x_1,\, k).$$

Since $x_0 = 0$,

$$\operatorname{sn}^{-1}(x_n,\, k) = n\operatorname{sn}^{-1}(x_1,\, k) = K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}$$

as the condition of porism.

Hence

$$x_1 = \operatorname{sn}\left\{\frac{K}{n},\, k\right\} = \frac{t_1e^{\omega/2}}{\sqrt{(1+t_1{}^2e^{\omega})}}$$

$$\therefore \quad \operatorname{tn}\left\{\frac{K}{n}\right\} = \left\{\frac{c-b}{c+b}\right\}^{\frac{1}{2}} \frac{\sqrt{(\lambda+1)}+\sqrt{(\lambda-1)}}{\sqrt{2}}$$

or

$$\operatorname{tn}\left\{\frac{K}{n}\right\} = \frac{c\sqrt{(b^2-a^2)}+b\sqrt{(c^2-a^2)}}{a(c+b)}.$$

Obviously, this cannot be the simplest condition for the porism of an n-agon in general, unless n be prime.

n = number of sides, d = degree of the rational porism in a, b, c

n	d	n	d	n	d	n	d	n	d	n	d
3	1	45	216	87	840	129	1848	171	3240	213	5040
4	2	46	264	88	960	130	2016	172	3696	214	5724
5	3	47	276	89	990	131	2145	173	3741	215	5544
6	4	48	256	90	864	132	1920	174	3360	216	5184
7	6	49	294	91	1008	133	2160	175	3600	217	5760
8	8	50	300	92	1056	134	2244	176	3840	218	5940
9	9	51	288	93	960	135	1944	177	3480	219	5328
10	12	52	336	94	1104	136	2304	178	3960	220	5760
11	15	53	351	95	1080	137	2346	179	4005	221	6048
12	16	54	324	96	1024	138	2112	180	3456	222	5472
13	21	55	360	97	1176	139	2415	181	4095	223	6216
14	24	56	384	98	1176	140	2304	182	4032	224	6144
15	24	57	360	99	1080	141	2208	183	3720	225	5400
16	32	58	420	100	1200	142	2520	184	4224	226	6384
17	36	59	435	101	1275	143	2520	185	4104	227	6441
18	36	60	384	102	1152	144	2304	186	3840	228	5760
19	45	61	465	103	1326	145	2520	187	4320	229	6555
20	48	62	480	104	1344	146	2664	188	4416	230	6336
21	48	63	432	105	1152	147	2352	189	3888	231	5760
22	60	64	512	106	1404	148	2736	190	4320	232	6720
23	66	65	504	107	1431	149	2775	191	4560	233	6786
24	64	66	480	108	1296	150	2400	192	4096	234	6048
25	75	67	561	109	1485	151	2850	193	4656	235	6624
26	84	68	576	110	1440	152	2880	194	4704	236	6960
27	81	69	528	111	1368	153	2592	195	4032	237	6240
28	96	70	576	112	1536	154	2880	196	4704	238	6912
29	105	71	630	113	1596	155	2880	197	4851	239	7140
30	96	72	576	114	1440	156	2688	198	4320	240	6144
31	120	73	666	115	1584	157	3081	199	4950	241	7260
32	128	74	684	116	1680	158	3120	200	4800	242	7260
33	120	75	600	117	1512	159	2808	201	4488	243	6561
34	144	76	720	118	1740	160	3072	202	5100	244	7440
35	144	77	720	119	1728	161	3168	203	5040	245	7056
36	144	78	672	120	1536	162	2916	204	4608	246	6720
37	171	79	780	121	1815	163	3321	205	5040	247	7560
38	180	80	768	122	1860	164	3360	206	5304	248	7680
39	168	81	729	123	1680	165	2880	207	4752	249	6888
40	192	82	840	124	1920	166	3444	208	5376	250	7500
41	210	83	861	125	1875	167	3486	209	5400	251	7875
42	192	84	768	126	1728	168	3072	210	4608	252	6912
43	231	85	864	127	2016	169	3549	211	5565	253	7920
44	240	86	924	128	2048	170	3456	212	5616	254	8064

7

All porisms are homogeneous in a, b, c. It is noticed that $C(3)$ is of degree 1, $C(5)$ of degree 3, $C(7)$ of degree 6 and in general $C(p)$ is of degree $\frac{1}{8}(p^2-1)$ for prime p. If n be a product of two distinct primes p and q, each different from 2, then the porisms corresponding to p and q intrude as irrelevant factors. Hence the chaste porism $C(pq)$ will be homogeneous in a, b, c and of degree $\frac{1}{8}[(p^2q^2-1)-(p^2-1)-(q^2-1)]$ or $\frac{1}{8}(p^2-1)(q^2-1)$. However, if n is a prime square, the degree of $C(p^2)$ is $\frac{1}{8}[(p^4-1)-(p^2-1)] = \frac{1}{8}p^2(p^2-1)$. Now since $C(p)$ is of degree $\frac{1}{8}(p^2-1)$, $K(2p)$ will be of degree $\frac{1}{2}(p^2-1)$, and hence the transformation T_p will multiply the degree of $C(p)$ by $\frac{1}{2}(p^2-1).2+1$ or by p^2. Hence T_p onC_p is of degree $p^2.\frac{1}{8}(p^2-1)$ or $\frac{1}{8}p^2(p^2-1)$. It follows that $C(p^2)$, given by T_p on $C(p)$, contains no irrelevant factors.

By similar cousiderations, it follows that $C(pqr)$, where $p, q, r > 2$ and prime, would be of degree

$$\tfrac{1}{8}[(p^2q^2r^2-1)-(p^2-1)-(q^2-1)-(r^2-1)-(p^2-1)(q^2-1)-(q^2-1)(r^2-1) \\ -(r^2-1)(p^2-1)] = \tfrac{1}{8}(p^2-1)(q^2-1)(r^2-1).$$

Now T_p on $C(q)$ is of degree $\frac{1}{8}p^2(q^2-1)$ and hence contains a factor of degree $\frac{1}{8}(q^2-1)$ which corresponds to $C(q)$. Hence

$$T_p \text{ on } C(q) \equiv C(q)\, C(pq).$$

Similarly

$$T_q \text{ on } C(p) \equiv C(p)\, C(pq),$$

We came across this phenomenon when discussing $C(15)$ and $C(21)$. But T_p on $C(pq)$ is of degree $\frac{1}{8}p^2(p^2-1)(q^2-1)$ and hence contains no irrelevant factors. In general, it is found that

$$T_p^{\alpha-1}\, T_q^{\beta-1}\, C(pqr) \equiv C(p^\alpha q^\beta r),$$

and contains no irrelevant factors.

As the transformation T_2 multiplies the dimension by 4, the counting of dimensions for even porisms presents no problem whatsoever. The table given on page 103 gives the dimensions of porisms for an n-agon for $3 \leq n \leq 254$. An interesting fact, capable of a simple proof, emerges from the table. If the dimension of a porism is odd, the polygon has p^r sides, where p is prime and r a positive integer.

The full invariant forms for the porisms in terms of Δ, Δ', Θ, Θ', may be obtained by the transformation

$$\frac{\Delta}{R^2} = \frac{\Theta}{2R^2+r^2-d^2} = \frac{\Theta'}{R^2+2r^2-d^2} = \frac{\Delta'}{r^2},$$

but like Chaundy, I am one of those who prefer to see the porisms exhibited in circular terms.

DEPARTMENT OF MATHEMATICS,
PRESIDENCY COLLEGE,
CALCUTTA.

References

Cayley, *Phil. Mag.*, V, (1853), pp. 281-284.

Cayley, *Phil. Mag.*, VI, (1853), pp. 99-102.

Cayley, *Phil. Trans. Roy Soc*, 1861 p. 225.

Chaundy, T. W, *Proc. Lond. Math Soc.*, series 2, XXII, 1923, pp- 104-123.

Chaundy, T. W., *Proc Lond Math Soc.*, series 2, XXV, 1926, pp. 17-44.

Clifford, W. K., *Proc. Lond. Math Soc.*, VII, Nos. 90-91, pp. 29-38

Clifford, W. K, *Proc Lond Math. Soc.*, VII, Nos. 102-103, pp. 225-233.

Clifford, W. K., *Camb Phil. Soc Proc.*, 1868, pp. 120-123.

Darboux. G, *Compte Rendus de l' Academie des Sciences*, XC, 1880.

Darboux. G., *Principes de géométrie analytique*, livre 3, 1917, pp. 250-87. Paris.

Greenhill, *Elliptic Functions*, Ch. 4, 1892.

Gulasekharam, F. V. H., *Math. Gazette*, Feb. 1941, p. 28.

Halphen, *Traité de fonctions elliptiques*, tome 2, p 370.

Richelot, *Crelle*, Vol. 38, p. 353

Rosanes and Pasch, *Crelle*, Vol. 64, § 9, pp. 163-164.

Steiner. *Crelle*, Vol. 2, 1827, p. 289.

Titchmarsh, E. C. *Messenger of Mathematics*, Vol 52, 1922, p 42.

Weill Mathieu *Bulletin de la Société Mathematique*, Vol. 29, 1901, pp. 199 208.

Wolstenholme, *Proc Lond. Math Soc.*, Vol. 8, p 136.

ON A NEW FUNCTION-THEORETIC METHOD OF SOLVING THE TORSION PROBLEM FOR SOME BOUNDARIES

By

S. Ghosh

(*Received September 10, 1947*)

1. It is well-known (Muschelisvili, 1929) that the problem of torsion of an isotropic elastic beam can be solved, when we know the mapping function which makes a conformal representation of the cross-section of the beam on the unit circle. In many cases, where the boundary of the cross-section consists partly of a straight line, this mapping function is very complicated, while we can find in a simple form the function which maps the cross-section conformally on a semi-circular area, the straight boundary of the cross-section corresponding to the bounding diameter. The method of determining the complex torsion function, discussed in this paper, relates to such cross-sections. The plane of the cross-section of the beam is taken to be the z-plane ($z = x+iy$), and the straight part of the boundary is taken to lie on the real axis in that plane. The plane of the semi-circular area, on which the cross-section is mapped conformally, is taken to be the ζ-plane ($\zeta = \xi+i\eta$), its radius unity and its bounding diameter on the real axis in that plane. The torsion problem is then reduced to the determination of a function of ζ, which is analytic within the semi-circular area, and is real on the bounding diameter, and whose imaginary part takes up prescribed values on the semi-circumference. By Schwarz's principle of reflection, this function is continued analytically to the other half of the unit circle, so that the problem becomes one of determination of a function, which is analytic within the unit circle, and whose imaginary part takes up prescribed values on the circumference. This function is then determined with the help of Schwarz's formula.

2. Let Γ be a closed curve in the z-plane, which is symmetrical about the real axis, and C the part of Γ above the real axis. Let

$$z = w(\zeta) \tag{2.1}$$

give the conformal representation of the region within Γ, on the unit circle in the ζ-plane, so that the region R in the z-plane bounded by C and the real axis is mapped conformally on the upper half of the unit circle in the ζ-plane.

Let R denote the cross-section of the beam. Then, in the torsion problem, we are to find a function

$$f(z) \equiv \phi+i\psi, \tag{2.2}$$

analytic in R, such that its imaginary part ψ takes up the value $\frac{1}{2}(x^2+y^2)$ on the boundary. If we put

$$F(z) \equiv \Phi+i\Psi = f(z)-\tfrac{1}{2}iz^2, \tag{2.3}$$

then the function $F(z)$ is analytic in R, and its imaginary part takes up the value y^2 on the boundary. Now, $y = 0$ on the part of the boundary which lies on the real axis, so that the function $F(z)$ is analytic in R, real on the real axis, and its imaginary part takes up the value y^2 on the boundary. By Schwarz's principle of reflection, this function can be continued analytically to the part of the region within Γ, lying below the real axis, the value of the function at the point $(x, -y)$ being taken to be the complex conjugate of its value at the point (x, y). Therefore, on the lower half of Γ, the imaginary part of the function takes up the value $-y^2$.

If ζ be the point on the circumference γ of the unit circle, corresponding to the point z on Γ, then we have for a point on Γ,

$$y^2 = -\tfrac{1}{4}[\omega(\zeta) - \bar{\omega}(1/\zeta)]^2, \tag{2.4}$$

since $\bar{\zeta} = 1/\zeta$ on γ.

Let

$$F[\omega(\zeta)] = G(\zeta). \tag{2.5}$$

Then the function $G(\zeta)/i$ is analytic within the unit circle in the ζ-plane, and its real part takes up the value

$$-\tfrac{1}{4}[\omega(\zeta) - \bar{\omega}(1/\zeta)]^2 \tag{2.6}$$

on the upper half α of γ, and the value

$$\tfrac{1}{4}[\omega(\zeta) - \bar{\omega}(1/\zeta)]^2 \tag{2.7}$$

on the lower half β of γ. Since an additive constant in the complex torsion function is immaterial, we can use Schwarz's formula in the modified from (Ghosh, 1947, p. 2), and write

$$G(\zeta) = -\frac{1}{4\pi}\int_\alpha \frac{[\omega(t) - \bar{\omega}(1/t)]^2}{t - \zeta}\,dt + \frac{1}{4\pi}\int_\beta \frac{[\omega(t) - \bar{\omega}(1/t)]^2}{t - \zeta}\,dt. \tag{2.8}$$

The torsional rigidity of the beam is μN_0, where

$$N_0 = \iint \left(x^2 + y^2 - x\frac{\partial\psi}{\partial x} - y\frac{\partial\psi}{\partial y}\right)dx\,dy, \tag{2.9}$$

the integration being taken over the cross-section of the beam.

Substituting

$$\psi = \Psi + \tfrac{1}{2}(x^2 - y^2)$$

from (2.3), we get

$$N_0 = N_{01} + N_{02}, \tag{2.10}$$

where

$$N_{01} = 2\iint y^2\,dx\,dy, \tag{2.11}$$

and

$$N_{02} = -\iint \left(x\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y}\right)dx\,dy = -\int \Phi(x\,dx + y\,dy) = -\tfrac{1}{2}\mathcal{R}\int F(z)\,d(z\bar{z}), \tag{2.12}$$

$\mathcal{R}$ denoting the real part of the line integral which is taken along the boundary of the cross-section. Hence

$$N_{02} = -\tfrac{1}{2}\mathcal{R}\int G(\zeta)d[\omega(\zeta)\bar{\omega}(\bar{\zeta})]$$

taken along the boundary of the semi-circular area in the ζ-plane. We can write this in the form

$$N_{02} = -\tfrac{1}{2}\mathcal{R}\int_{\alpha} G(\zeta)d[\omega(\zeta)\omega(1/\zeta)] - \tfrac{1}{2}\int_{-1}^{1} G(\xi)d[\omega(\xi)\bar{\omega}(\xi)], \tag{2.13}$$

the second integral on the right being evidently real.

3. As an example, let us consider a beam whose cross-section is a semi-circular area bounded by the circle $x^2+y^2 = 1$, $y \geq 0$, and the diameter $y = 0$, $-1 \leqq x \leqq 1$.

Here (2.1) is the identical transformation

$$z = \omega(\zeta) = \zeta. \tag{3.1}$$

Replacing ζ by z in (2.8) and observing that $G(z)$ is identical with $F(z)$, we have

$$F(z) = -\frac{1}{4\pi}\int_{\alpha}\left(t-\frac{1}{t}\right)^2\frac{dt}{t-z} + \frac{1}{4\pi}\int_{\beta}\left(t-\frac{1}{t}\right)^2\frac{dt}{t-z}, \tag{3.2}$$

where α and β are now the upper and the lower halves respectively of the circumference of the unit circle in the z-plane.

Now

$$\int\left(t-\frac{1}{t}\right)^2\frac{dt}{t-z} = \frac{1}{2}t^3+zt+\frac{(z^2-1)^2}{z^2}\log(t-z)-\frac{\log t}{z^2}+\frac{1}{zt}.$$

As we describe α in the positive sense, $|t| = 1$ and am t changes from 0 to π. Also, as we describe β in the positive sense, $|t| = 1$ and am t changes from $-\pi$ to 0. Thereofre

$$\int_{\alpha}\left(t-\frac{1}{t}\right)^2\frac{dt}{t-z} = -2z+\frac{(z^2-1)^2}{z^2}\left\{i\pi+\log\frac{1+z}{1-z}\right\}-\frac{i\pi}{z^2}-\frac{2}{z},$$

and

$$\int_{\beta}\left(t-\frac{1}{t}\right)^2\frac{dt}{t-z} = 2z+\frac{(z^2-1)^2}{z^2}\left\{i\pi+\log\frac{1-z}{1+z}\right\}-\frac{i\pi}{z^2}+\frac{2}{z}.$$

Substituting in (3.2), we get

$$F(z) = \frac{1}{2\pi}\left[\frac{2(z^2+1)}{z}+\frac{(z^2-1)^2}{z^2}\log\frac{1-z}{1+z}\right]. \tag{3.3}$$

The complex torsion function is then calculated from (2.3), and is found to be

$$f(z) = \frac{1}{2\pi}\left[\pi iz^2+\frac{2(z^2+1)}{z}+\frac{(z^2-1)^2}{z^2}\log\frac{1-z}{1+z}\right], \tag{3.4}$$

which agrees with the result given by Love (1927, p. 319) and cited by him as due to Greenhill. The same result has also been obtained by Sokolnikoff and Sokolnikoff (1938), who have used the method of Muschelisvili.

N_0 is calculated from (2.10), (2.11) and (2.12). We see at once that

$$N_{01} = \frac{\pi}{4}. \tag{3.5}$$

On the circumference, $z\bar{z}$ is constant, and on the bounding diameter (which lies on the real axis), the integral in (2.12) is real. Therefore, we have

$$\begin{aligned} N_{02} &= -\frac{1}{2\pi}\int_{-1}^{1}\left[2(x^2+1)+\frac{(x^2-1)^2}{x}\log\frac{1-x}{1+x}\right]dx \\ &= -\frac{8}{3\pi}+\frac{1}{2\pi}\int_{-1}^{1}\left(x^3-2x+\frac{1}{x}\right)\log\frac{1+x}{1-x}\,dx. \end{aligned} \tag{3.6}$$

But, it can be easily proved that

$$\int_{-1}^{1}(x^3-2x)\log\frac{1+x}{1-x}\,dx = -\frac{8}{3}$$

and

$$\int_{-1}^{1}\frac{1}{x}\log\frac{1+x}{1-x}\,dx = \frac{\pi^2}{2}.$$

Therefore

$$N_{02} = -\frac{4}{\pi}+\frac{\pi}{4}, \tag{3.7}$$

and consequently,

$$N_0 = \frac{\pi^2-8}{2\pi}. \tag{3.8}$$

This is in agreement with the result given by Stevenson (1938, p. 193).

4. As a second example, let us consider a beam whose cross-section is a semi-loop of the lemniscate of Bernoulli. We know that

$$z = \omega(\zeta) = a\sqrt{1+\zeta} \tag{4.1}$$

gives a conformal representation of the region within a complete loop of Bernoulli's lemniscate on the unit circle, and at the same time, a conformal representation of the region bounded by the upper half of the lemniscate and the real axis in the z-plane, on the upper half of the unit circle.

For a point on the circumference of the unit circle, we have

$$[\omega(t)-\bar{\omega}(1/t)]^2 = a^2\left[2+\left(t+\frac{1}{t}\right)-\frac{2(1+t)}{\sqrt{t}}\right]. \tag{4.2}$$

Substituting in (2.8), we get

$$G(\zeta) = -\frac{a^2}{4\pi}\int\left[2+\left(t+\frac{1}{t}\right)-\frac{2(1+t)}{\sqrt{t}}\right]\frac{dt}{t-\zeta}+\frac{a^2}{4\pi}\int_{\beta}\left[2+\left(t+\frac{1}{t}\right)-\frac{2(1+t)}{\sqrt{t}}\right]\frac{dt}{t-\zeta}. \tag{4.3}$$

Observing that am t changes from 0 to π and from $-\pi$ to 0 respectively, as we describe α and β respectively, in the positive sense, while $|t| = 1$ on both, we prove easily that

$$\int_\alpha \frac{dt}{t-\zeta} = i\pi + \log\frac{1+\zeta}{1-\zeta},$$

$$\int_\beta \frac{dt}{t-\zeta} = i\pi + \log\frac{1-\zeta}{1+\zeta},$$

$$\int_\alpha \left(t+\frac{1}{t}\right)\frac{dt}{t-\zeta} = -\frac{i\pi}{\zeta} + \frac{1+\zeta^2}{\zeta}\left\{i\pi + \log\frac{1+\zeta}{1-\zeta}\right\},$$

$$\int_\beta \left(t+\frac{1}{t}\right)\frac{dt}{t-\zeta} = -\frac{i\pi}{\zeta} + \frac{1+\zeta^2}{\zeta}\left\{i\pi + \log\frac{1-\zeta}{1+\zeta}\right\},$$

$$\int_\alpha \frac{1+t}{\sqrt{t}}\cdot\frac{dt}{t-\zeta} = 2(i-1) + \frac{1+\zeta}{\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\cdot\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}},$$

$$\int_\beta \frac{1+t}{\sqrt{t}}\cdot\frac{dt}{t-\zeta} = 2(1+i) + \frac{1+\zeta}{\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\cdot\frac{1-\sqrt{\zeta}}{1+\sqrt{\zeta}}.$$

Substituting in (4.3) and simplifying, and omitting an irrelevant additive constant, we get

$$G(\zeta) = -\frac{a^2}{2\pi}\cdot\frac{(1+\zeta)^2}{\zeta}\log\frac{1+\zeta}{1-\zeta} + \frac{a^2}{\pi}\cdot\frac{1+\zeta}{\sqrt{\zeta}}\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}. \tag{4.4}$$

N_0 is calculated from (2.10), (2.11) and (2.13). We see at once that

$$N_{01} = a^4\left(\frac{\pi}{8} - \frac{1}{3}\right). \tag{4.5}$$

To evaluate the first integral in (2.13), we observe that

$$\omega(\zeta)\bar{\omega}(1/\zeta) = \frac{a^2(1+\zeta)}{\sqrt{\zeta}}.$$

Substituting in the integral, we can prove that

$$\int G(\zeta)d[\omega(\zeta)\bar{\omega}(1/\zeta)] = \frac{a^4}{6\pi}\left[-\frac{(1+\zeta)^3}{\zeta\sqrt{\zeta}}\log\frac{1+\zeta}{1-\zeta} + \frac{3+2\zeta+3\zeta^2}{\zeta}\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}} + \frac{2(1+\zeta)}{\sqrt{\zeta}}\right],$$

so that

$$\int_\alpha G(\zeta)d[\omega(\zeta)\bar{\omega}(1/\zeta)] = -\frac{2a^4}{3\pi}\left(1 + 2\log 2 + \frac{i\pi}{2}\right). \tag{4.6}$$

To evaluate the second integral in (2.13), we observe that

$$\omega(\xi)\bar{\omega}(\xi) = a^2(1+\xi),$$

so that

$$\int_{-1}^{1} G(\xi)d[\omega(\xi)\bar{\omega}(\xi)] = -\frac{a^4}{2\pi}\int_{-1}^{1}\frac{(1+\xi)^2}{\xi}\log\frac{1+\xi}{1-\xi}\,d\xi + \frac{a^4}{\pi}\int_{-1}^{1}\frac{1+\xi}{\sqrt{\xi}}\log\frac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\,d\xi. \tag{4.7}$$

Now, it can be easily proved that

$$\int_{-1}^{1}(2+\xi)\log\frac{1+\xi}{1-\xi}\,d\xi = 2$$

and

$$\int_{-1}^{1}\frac{1}{\xi}\log\frac{1+\xi}{1-\xi}\,d\xi = \frac{\pi^2}{2}.$$

Therefore

$$\int_{-1}^{1}\frac{(1+\xi)^2}{\xi}\log\frac{1+\xi}{1-\xi}\,d\xi = \left(\frac{\pi^2}{2}+2\right). \tag{4.8}$$

Further, we can show that

$$\int_{-1}^{1}\frac{1+\xi}{\sqrt{\xi}}\log\frac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\,d\xi = \frac{2}{3}(2+4\log 2+\pi). \tag{4.9}$$

Substituting from (4.8) and (4.9) in (4.7), we get

$$\int_{-1}^{1} G(\xi)d[\omega(\xi)\bar{\omega}(\xi)] = \frac{2a^4}{3\pi}\left(-\tfrac{3}{8}\pi^2+\pi+\tfrac{1}{2}+4\log 2\right). \tag{4.10}$$

From (2.13), (4.6) and (4.10), we then get

$$N_{03} = (a^4/\pi)\left(\tfrac{1}{8}\pi^2-\tfrac{1}{3}\pi+\tfrac{1}{6}-\tfrac{8}{3}\log 2\right). \tag{4.11}$$

Finally, (4.5) and (4.11) give

$$N_0 = N_{01}+N_{02} = (a^4/\pi)\left(\tfrac{1}{4}\pi^2-\tfrac{2}{3}\pi+\tfrac{1}{6}-\tfrac{8}{3}\log 2\right). \tag{4.12}$$

This is in agreement with Stevenson's result (1938, p. 217).

References

Ghosh, S., (1947), *Bull. Cal. Math. Soc.*, **39**, 1

Love, A. E. H., (1927), *Mathematical theory of Elasticity (4th. ed.)*, Cambridge.

Muschelisvili, N. I., (1929) *Atti della reale accademia nazionale dei Lincei* (6), **9**, 295.

Sokolnikoff, I. S. and Sokolnikoff, E. S., (1938), *Bull. Am. Math. Soc.* **44**, 384.

Stevenson, A. C., (1938), *Phil. Trans. Roy. Soc., Lond.* (A), **237**, 161.

DIRECT DETERMINATION OF STRESSES IN THIN ELASTIC PLATES HAVING CAVITIES OF DIFFERENT SHAPES

By

Bibhutibhusan Sen

(*Received September 30, 1947*)

1. Introduction

A new direct method of finding stresses in large thin plates with small holes having different curvilinear boundaries has been discussed in a recent paper (Sen, 1946). It has been supposed in that paper that the plates have a distribution of all round tension at great distances from the cavities, and that no external forces are applied to their internal boundaries. In the present paper, the same method is employed to solve similar problems of large thin plates with holes of different shapes having hydrostatic thrusts on their inner edges. In these cases, stresses at great distances from these edges are assumed to be zero. The cavities considered are those having boundaries in the form of (1) an inverse of an ellipse, (2) a loop of a lemniscate of Bernoulli, and (3) an approximate square with rounded corners.

2. Method of Solution

Suppose

$$z = f(\zeta), \tag{2.1}$$

where

$$z = x+iy, \quad \text{and} \quad \zeta = \xi+i\eta.$$

If the boundaries of the cavities be obtained by putting either $\xi =$ constant or $\eta =$ constant, it is found that the solution of the problems under consideration is much simplified when the *mean* plane stresses $\widehat{\xi\xi}$, $\widehat{\xi\eta}$, and $\widehat{\eta\eta}$ are known in terms of ξ and η. In the paper mentioned above, it has been shown that when there are no body forces, these stress components satisfying the equations of equilibrium and compatibility are given by the relations

$$\frac{8\widehat{\xi\xi}}{h^2} = \frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \eta} - \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \xi} + \frac{4\odot}{h^2} + F, \tag{2.2a}$$

$$\frac{8\widehat{\eta\eta}}{h^2} = \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \eta} + \frac{4\odot}{h^2} - F, \tag{2.2b}$$

$$\frac{8\widehat{\xi\eta}}{h^2} = -\frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \eta} - G, \tag{2.2c}$$

where

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2, \quad r^2 = x^2+y^2, \quad \odot = \widehat{\xi\xi}+\widehat{\eta\eta},$$

and F and G are conjugate harmonic functions.

Alternatively, these results can be expressed as

$$\frac{8\widehat{\xi\xi}}{h^4} = \frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}\left(\frac{\odot}{h^2}\right) - \frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{\odot}{h^2}\right) + \frac{4\odot}{h^4} - \odot\left[\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{h^2}\right) - \frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{1}{h^2}\right)\right] + \frac{F}{h^2}, \tag{2.3a}$$

$$\frac{8\widehat{\eta\eta}}{h^4} = \frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{\odot}{h^2}\right) - \frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}\left(\frac{\odot}{h^2}\right) + \frac{4\odot}{h^4} - \odot\left[\frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{1}{h^2}\right) - \frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{h^2}\right)\right] - \frac{F}{h^2}, \tag{2.3b}$$

$$\frac{8\widehat{\xi\eta}}{h^4} = -\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\xi}\left(\frac{\odot}{h^2}\right) - \frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\eta}\left(\frac{\odot}{h^2}\right) - \odot\left[\frac{\partial r^2}{\partial\eta}\frac{\partial}{\partial\xi}\left(\frac{1}{h^2}\right) + \frac{\partial r^2}{\partial\xi}\frac{\partial}{\partial\eta}\left(\frac{1}{h^2}\right)\right] - \frac{G}{h^2}. \tag{2.3c}$$

The results (2.2) and (2.3) show that the stress components $\widehat{\xi\xi}$, $\widehat{\xi\eta}$ and $\widehat{\eta\eta}$ can be expressed explicitly in terms of the harmonic function $\odot$ and two conjugate functions F and G. If, for example, $\widehat{\xi\eta}$ is given on the boundary $\xi =$ constant, we can choose a tentative harmonic function for $\odot$ involving certain constant coefficients, and hence, determine an expression for G from either (2.2c) or (2.3c). The relation between two conjugate functions gives the expression for F and, then, from the known values of $\widehat{\xi\xi}$ on the boundary, the assumed constants in $\odot$ (which will also occur in F) can be determined either from (2.2a) or (2.3a). $\odot$, F and G being thus found, the stress components will be exactly known. It should be noted that the assumed value of $\odot$ which is a harmonic function may either be expressed in an infinite series or in a closed form.

3. A CAVITY IN THE FORM OF AN INVERSE OF AN ELLIPSE

If we use the transformation

$$x+iy = c\sec(\xi+i\eta) \tag{3.1}$$

we get

$$x = \frac{2c\cosh\eta\cos\xi}{\cosh 2\eta+\cos 2\xi}, \qquad y = \frac{2c\sinh\eta\sin\xi}{\cosh 2\eta+\cos 2\xi},$$

$$r^2 = x^2+y^2 = \frac{2c^2}{\cosh 2\eta+\cos 2\xi},$$

and

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2 = \frac{2c^2(\cosh 2\eta-\cos 2\xi)}{(\cosh 2\eta+\cos 2\xi)^2}. \tag{3.2}$$

It can be easily seen that $\eta=\beta$ (a constant) is the inverse of an ellipse with respect to the centre. Let there be a hydrostatic thrust of magnitude P on the inner boundary of the hole given by $\eta=\beta$. We assume in this case

$$\odot = B(1+\cosh 2\eta\cos 2\xi) \tag{3.3}$$

where B is a constant. We find from the assumed value of $\odot$ that $\odot \to 0$, when $\eta \to 0$ and $\xi \to \pm\pi/2$, that is, when $r \to \infty$, and also that $\odot$ is a harmonic function.

Since $\widehat{\xi\eta} = 0$ when $\eta = \beta$, we have from (2.2c)

$$[G]_{\eta=\beta} = -\left[\frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \xi} + \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \eta}\right]_{\eta=\beta}$$

$$= -\frac{8Bc^2 \sinh 2\beta \sin 2\xi}{\cosh 2\beta + \cos 2\xi}. \tag{3.4}$$

The above value of G on the boundary will be obtained if we take

$$F + iG = -8Bc^3 \sinh 2\beta \frac{i \sin(\xi + i\eta)}{\cos(\xi + i\eta)}. \tag{3.5}$$

which gives

$$F = \frac{8Bc^2 \sinh 2\beta \sinh 2\eta}{\cosh 2\eta + \cos 2\xi}. \tag{3.6}$$

Substituting this value of F and the value of $\odot$ as given in (3.3) into (2.2b), we find that the condition $[\widehat{\eta\eta}]_{\eta=\beta} = -P$ is satisfied if

$$B = 2P \operatorname{cosech}^2 2\beta. \tag{3.7}$$

Thus we have

$$\odot = 2P \operatorname{cosech}^2 2\beta [1 + \cosh 2\eta \cos 2\xi],$$

$$F = \frac{16Pc^3 \operatorname{cosech} 2\beta \sinh 2\eta}{\cosh 2\eta + \cos 2\xi},$$

$$G = -\frac{16Pc^3 \operatorname{cosech} 2\beta \sin 2\xi}{\cosh 2\eta + \cos 2\xi}. \tag{3.8}$$

4. A CAVITY IN THE FORM OF A LOOP OF LEMNISCATE OF BERNOULLI

Let z and ζ be connected by the relation

$$(z + c)(z - c) = c^2 e^{2\zeta}, \tag{4.1}$$

where c is a constant. In cartesian co-ordinates x and y, we have the relations

$$(x^2 - y^2 - c^2)^2 + 4x^2y^2 = c^4 e^{4\xi},$$
$$(x^2 - y^2 - c^2) = 2xy \cot 2\eta, \tag{4.2}$$

so that the curve $\xi = 0$ becomes a lemniscate of Bernoulli of which the polar equation is

$$r^2 = 2c^2 \cos 2\theta. \tag{4.3}$$

We shall consider the loop of the curve that lies in the region

$$-\infty \leq \xi \leq 0, \quad -\pi/2 \leq \eta \leq \pi/2.$$

In the present case we get

$$\frac{1}{h^2} = \frac{c^2 e^{4\xi}}{[1 + e^{4\xi} + 2e^{2\xi}\cos 2\eta]^{\frac{1}{2}}}, \tag{4.4}$$

$$r^2 = c^2[1 + e^{4\xi} + 2e^{2\xi}\cos 2\eta]^{\frac{1}{2}}. \tag{4.5}$$

From these results we find that

$$\left[\frac{\partial r^2}{\partial \xi}\right]_{\xi=0} = 2c^2 \cos\eta, \quad \left[\frac{\partial r^2}{\partial \eta}\right]_{\xi=0} = -2c^2 \sin\eta, \quad \left[\frac{1}{h^2}\right]_{\xi=0} = \frac{c^2}{2}\sec\eta \tag{4.6}$$

Assuming the boundary of the cavity to be given by $\xi = 0$, we are now to choose a harmonic function $\odot$ which tends to zero at an infinite distance, that is, when $\xi \to \infty$. A simple expression for $\odot$ satisfying the above condition is given by

$$\odot = B\, e^{-2\xi} \cos 2\eta, \tag{4 7}$$

where B is a constant, to be determined.

If we now assume that there is only hydrostatic thrust of magnitude P on the inner boundary, we shall have

$$\widehat{\xi\eta} = 0, \quad \text{and} \quad \widehat{\xi\xi} = -P \quad \text{when} \quad \xi = 0.$$

From the first condition we obtain in virtue of the relation (2.2c)

$$[G]_{\xi=0} = -\left[\frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \xi} + \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \eta}\right]_{\xi=0} = 4Bc^2 \sin\eta. \tag{4.8}$$

From the second condition we have on using the relation (2.2a)

$$-\frac{4Pc^2}{\cos\eta} = \left[\frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \eta} - \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \xi} + \frac{4\odot}{h^2} + F\right]_{\xi=0}$$

$$= -\frac{2Bc^2}{\cos\eta} + 8Bc^2\cos\eta + [F]_{\xi=0}.$$

This equation is satisfied if

$$B = 2P, \quad \text{and} \quad [F]_{\xi=0} = -8Bc^2\cos\eta. \tag{4 9}$$

Expressions for F and G on the boundary $\xi=0$ as given in (4.9) and (4.8) will be obtained if we take

$$F + iG = -4Bc^2\left[e^{-(\xi+i\eta)} + \cosh(\xi + i\eta)\right]. \tag{4.10}$$

Thus we get

$$\odot = 2P\, e^{-2\xi} \cos 2\eta,$$

$$F = -8Pc^2\left[e^{-\xi} + \cosh\xi\right]\cos\eta,$$

$$G = 8Pc^2\left[e^{-\xi} - \sinh\xi\right]\sin\eta. \tag{4.11}$$

5. A CAVITY IN THE FORM OF AN APPROXIMATE SQUARE WITH ROUNDED CORNERS

Assuming

$$z = e^{\zeta} + c e^{-3\zeta},$$

we get

$$x = e^{\xi}\cos\eta + c e^{-3\xi}\cos 3\eta,$$

$$y = e^{\xi}\sin\eta - c e^{-3\xi}\sin 3\eta, \tag{5.1}$$

so that the boundary $\xi = 0$ gives an approximate square. We have in the present case

$$\frac{1}{h^2} = e^{2\xi} + 9c^2 e^{-6\xi} - 6c e^{-2\xi} \cos 4\eta, \tag{5.2}$$

$$r^2 = e^{2\xi} + c^2 e^{-6\xi} + 2c e^{-2\xi} \cos 4\eta. \tag{5.3}$$

Let us assume

$$\begin{aligned} \odot &= B\left\{1 - \mathbf{Re}\left[\frac{e^{\zeta} + 3c e^{-3\zeta}}{e^{\zeta} - 3c e^{-3\zeta}}\right]\right\} \\ &= \frac{6B[3c^2 e^{-6\xi} - c e^{-2\xi} \cos 4\eta]}{e^{2\xi} - 6c e^{-2\xi} \cos 4\eta + 9c^2 e^{-6\xi}}, \end{aligned} \tag{5.4}$$

where B is a constant and **Re** denotes the real part. This value of $\odot$ gives

$$\frac{\odot}{h^2} = 6B[3c^2 e^{-6\xi} - c e^{-2\xi} \cos 4\eta] \tag{5.5}$$

From (5.2) and (5.3) we get

$$\left[\frac{\partial r^2}{\partial \xi}\right]_{\xi=0} = 2 - 6c^2 - 4c \cos 4\eta, \quad \left[\frac{\partial r^2}{\partial \eta}\right]_{\xi=0} = -8c \sin 4\eta, \tag{5.6}$$

$$\left[\frac{\partial}{\partial \xi}\left(\frac{1}{h^2}\right)\right]_{\xi=0} = 2 - 54c^2 + 12c \cos 4\eta, \quad \left[\frac{\partial}{\partial \eta}\left(\frac{1}{h^2}\right)\right]_{\xi=0} = 24c \sin 4\eta, \tag{5.7}$$

We assume that on the boundary $\xi = 0$, $\widehat{\xi\eta} = 0$. Hence from (2 3c) we get

$$\left[\frac{G}{h^2}\right]_{\xi=0} = -48Bc(1 + 3c^2) \sin 4\eta. \tag{5.8}$$

The above value of G/h^2 on the boundary is obtained if we assume

$$F + iG = 4B(1 + 3c^2)\frac{e^{\zeta} + 9c e^{-3\zeta}}{e^{\zeta} - 3c e^{-3\zeta}}. \tag{5.9}$$

This gives

$$\left[\frac{F}{h^2}\right]_{\xi=0} = 4B(1 + 3c^2)[(1 - 27c^2) + 6c \cos 4\eta]. \tag{5.10}$$

Since by assumption $\widehat{\xi\xi} = -P$ on the boundary $\xi = 0$, we have from (2.3*a*)

$$B = -2P. \tag{5.11}$$

With this value of B we get

$$\odot = \frac{12Pc[e^{-2\xi} \cos 4\eta - 3c e^{-6\xi}]}{e^{2\xi} + 9c^2 e^{-6\xi} - 6c e^{-2\xi} \cos 4\eta},$$

$$F = -8P(1 + 3c^2)\frac{e^{2\xi} - 27c^2 e^{-6\xi} + 6c e^{-2\xi} \cos 4\eta}{e^{2\xi} + 9c^2 e^{-6\xi} - 6c e^{-2\xi} \cos 4\eta},$$

$$G = 96P(1+3c^2)\frac{ce^{-2\xi}\sin 4\eta}{e^{2\xi}+9c^2e^{-6\xi}-6ce^{-2\xi}\cos 4\eta}. \tag{5.12}$$

BENGAL ENGINEERING COLLEGE,
SIBPORE.

Reference

Sen, B., (1946), *Philosophical Magazine* (Not yet published).

REMAINDERS IN QUADRATURE FORMULAE

By

M. C. Chakrabarti

(Communicated by N. M. Basu—Received December 9, 1947)

1. The object of quadrature formulae is to approximate a definite integral

$$\int_a^b f(x)dx$$

by means of an expression of the type

$$\sum_{\alpha=1}^{p} R_\alpha f(x_\alpha)$$

where the R's are constants and $x_1, x_2, \ldots, x_p$ are p values of x in the in the interval (a, b). If $f(x)$ be expressible in the form:

$$f(x) = \sum_{r=0}^{\infty} a_r x^r,$$

the error of approximation $L(f)$ can be written as

$$L(f) = \sum_{r=0}^{\infty} a_r \frac{b^{r+1}-a^{r+1}}{r+1} - \sum_{\alpha=1}^{p} R_\alpha \sum_{r=0}^{\infty} a_r x_\alpha^r$$

$$= \sum_{r=0}^{\infty} \left[a_r \left\{ \frac{b^{r+1}-a^{r+1}}{r+1} - \sum_{\alpha=1}^{p} R_\alpha x_\alpha^r \right\} \right].$$

Let

$$\frac{b^{r+1}-a^{r+1}}{r+1} - \sum_{\alpha=1}^{p} R_\alpha x_\alpha^r = 0 \text{ for } r = 0, 1, 2, \ldots, n-1.$$

and

$$\Omega = \frac{b^{n+1}-a^{n+1}}{n+1} - \sum_{\alpha=1}^{p} R_\alpha x_\alpha^n \neq 0.$$

Then

$$L(f) = \sum_{r=n}^{\infty} \left[a_r \left\{ \frac{b^{r+1}-a^{r+1}}{r+1} - \sum_{\alpha=1}^{p} R_\alpha x_\alpha^r \right\} \right]$$

$$= a_n \Omega + a_{n+1} (\quad) + \cdots \tag{1}$$

The term $a_n\Omega$ in (1) is known as the leading term of the error series and is usually given with the formulae to be used. The closeness of approximation is usually judged by its magnitude.

Let the remainder in a quadrature formula be expressible in the form.

$$L(f) = \sum_r A_r f(x_r) + \sum_s B_s f'(x_s) + \sum_t C_t f''(x_t) + \cdots \tag{2}$$

to a finite number of terms. The formula (2) will be of order n if $L(x^q) = 0$ for $q = 0, 1, 2, \ldots, n-1$ and $L(x^n) = E \neq 0$. A formula of order n is said to the *simplex* if for every function $f(x)$ possessing derivatives up to n-th order

$$L(f) = 0 \text{ implies } f^{(n)}(\xi) = 0$$

for some value ξ of x in the interval containing the points.

Daniel (1940) has shown that for formulae which are simplex, the remainder comes out in a simple form. Let

$$\frac{L(f)}{E} = \lambda, \; g(x) = f(x) - \lambda x^n$$

then since the formula is simplex $L(g)=0$ will imply that for some ξ in the interval $g^{(n)}(\xi) = 0$. Thus

$$0 = g^{(n)}(\xi) = f^{(n)}(\xi) - \lambda . n! = f^{(n)}(\xi) - \frac{L(f)}{E} n!$$

whence

$$L(f) = E . \frac{f^{(n)}(\xi)}{n!}.$$

Evidently the order of (1) is $n+1$ and

$$\Omega = E/(n+1).$$

In a recent paper Bickley (1939) has given a long list of quadrature formulae with the leading term in the error series. In the present paper it has been shown that most of these formulae are simplex so that the leading term in the error series is the actual error when the value of the derivative is taken at a suitable intermediate point. Besides, a a number of simple theorems are proved which will enable one to decide quickly whether a particular formula is simplex or not. In Section 2, the usual methods employed for judging whether a given formula is simplex or not are illustrated by a few examples.

2. Consider the following four formulae ·

(*i*) $L(f) = f(2) - f(0) - \{29f'(0) + 124f'(1) + 24f'(2) + 4f'(3) - f'(4)\}/90$

(*ii*) $L(f) = 3\{f(3) - f(2) - f(1) + f(0)\} - \{f'(3) + 3f'(2) - 3f'(1) - f'(0)\}$

(*iii*) $L(f) = f(1) - f(-1) - 2\{f'(0) + f'(1/\sqrt{2}) + f'(-1/\sqrt{2})\}/3$

(*iv*) $L(f) = f(4) - f(0) - 4\{2f'(0) - 4f'(1) + 5f'(2)\}/3$

In (*i*) the order is 6. Choose a polynomial of degree 5 agreeing with f at 0, 1, 2 and having the same derivative as f at 0, 1, 4. Consider $g = f - p$. $L(f) = 0$ implies $L(g) = 0$ which gives $24g'(2) + 4g'(3) = 0$ whence $g'(x) = 0$ for some valne ξ of x between 2 and 3. Hence $g'(x)$ vanishes at 6 distinct points and by successive applications of Rolle's Theorem we can show that $g^{(6)}(x)$, *i.e.*, $f^{(6)}(x)$ vanishes at least once in the interval. Therefore, the formula is simplex and

$$E = L(x^6) = L\{x(x-2)^2(x-4)(x-3)(x-1)\} = 1/8.$$

Therefore

$$L(f) = \frac{1}{8} . \frac{f^{(6)}(\xi)}{6!} = f^{(6)}(\xi)/90.$$

In *(ii)* the order is also 6. As before let $p(x)$ be a polynomial of degree 5 at most such that $f = p$ at $x = 2, 3$ and $f' = p'$ at $x = 0, 1, 2, 3$. Consider $g = f - p$. $L(f) = 0$ implies $L(g) = 0$ which in turn implies $-g(1) + g(0) = 0$, *i.e.*, $g'(x) = 0$ for some value ξ of x between 0 and 1. It will be found that $g'(x) = 0$ at $x = 0, \xi, 1, 2, \xi_1, 3$, *i.e.*, at 6 distinct points. Hence $g^{(6)}(x) = 0$ at least between 0 and 3, *i.e.*, $L(f)$ is simplex. And

$$E = L(x^6) = -24, \quad L(f) = -24\,\frac{f^{(6)}(\xi)}{6!} = -\frac{f^{(6)}(\xi)}{30}\,.$$

Formula *(iii)* is of order 5. And

$$f'(x) = -2(x^2 - \tfrac{1}{2})f'(0) + x\{x - (1/\sqrt{2})\}f'(-1/\sqrt{2})$$
$$+\, x\{x + (1/\sqrt{2})\}f'(1/\sqrt{2}) + x(x^2 - \tfrac{1}{2})f'(x,\, 0,\, -1/\sqrt{2},\, 1/\sqrt{2})$$

where $f'(x_1 x_2 \ldots x_n)$ denotes the divided difference of f' of order $n-1$. Integrating between the limits -1 and 1, we get

$$\begin{aligned}
L(f) &= f(1) - f(-1) - \tfrac{2}{3}\{f'(0) + f'(1/\sqrt{2}) + f'(-1/\sqrt{2})\}\\
&= \int_{-1}^{+1} x(x^2 - \tfrac{1}{2})f'(x,\, 0,\, 1/\sqrt{2},\, -1/\sqrt{2})dx\\
&= \left[\left(\frac{x^4}{4} - \frac{x^2}{4}\right) f'(x,\, 0,\, 1/\sqrt{2},\, -1/\sqrt{2})\right]_{-1}^{+1} - \int_{-1}^{+1} \frac{x^2}{4}\,(x^2 - 1)f'(x, x, 0, 1/\sqrt{2}, -1/\sqrt{2})dx\\
&= f^{(5)}(\xi)/360.
\end{aligned}$$

and the formula is simplex.

Formula *(iv)* is of order 4. We have, as before,

$$\begin{aligned}
L(f) &= \int_0^4 x(x-1)(x-2)f'(x,\, 0,\, 1,\, 2)dx\\
&= \left\{\int_0^3 + \int_3^4\right\} x(x-1)(x-2)f'(x,\, 0,\, 1,\, 2)dx\\
&= \int_0^3 x(x-2)[f'(x,\, 0,\, 2) - f'(0,\, 1,\, 2)]dx + f^{(4)}(\xi_1)(16 - 9/4)/6, \qquad (3 < \xi_1 < 4)\\
&= -\int_0^3 x^2\left(\frac{x}{3} - 1\right) f'(x,\, x,\, 0,\, 2)dx + f^{(4)}(\xi_1)(16 - 9/4)/6\\
&= \tfrac{1}{6}f^{(4)}(\xi_2).\tfrac{9}{4} + \tfrac{1}{6}f^{(4)}(\xi_1)(16 - \tfrac{9}{4}), \qquad 0 < \xi_2 < 3,\\
&= 16\,\frac{f^{(4)}(\xi)}{6} = (8/3)f^{(4)}(\xi), \qquad 0 < \xi < 4.
\end{aligned}$$

and the formula is simplex.

3. Often the question as to whether a formula simplex or not can be decided by an application of one of the following four theorems Theorems 1 and 4 are possibly new

but theorems 2 and 3 are given in text books in a different form and they have been restated here in a form suitable for our purpose.

Theorem 1. *Let*

$$L(f) = Af(b)+A'f(a)+\sum_{s=1}^{n} A_s f(x_s)-\sum_{s=1}^{n} B_s f'(x_s) \tag{3}$$

where $a \leq x_1 < x_2 \ldots < x_n \leq b$, A, A', A_s, B_s are constants and the coefficient of $f(b)$ is not zero. Then (3) will be simplex if (i) the order is $2n+1$, (ii) $a = x_1$, $b = x_n$ and the order is $2n-1$, (iii) $a = x_1$ and the order is $2n$, (iv) $a = x_1$, $b = x_n$, r of the B's are zero and the order is greater than or equal to $2n-r-1$.

Proof: (i) Let $p(x)$ be a polynomial of degree $2n$ at most such that (α) when $a < x_1$, $x_n < b$, $p(x) = f(x)$ for $x = a, x_1, x_2, \ldots x_n$ and $p'(x) = f'(x)$ for $x = x_1, x_2, \ldots, x_n$; ($\beta$) when $a = x_1$, $x_n < b$, $p(x) = f(x)$ for $x = x_1, x_2, \ldots, x_n$, $p'(x) = f'(x)$ for $x = x_1, x_2, \ldots, x_n$ and $p''(b) = f''(b)$; (γ) when $a = x_1$, $x_n = b$, $p(x) = f(x)$ for $x = x_1, x_2, \ldots, x_{n-1}$, $p' = f'$ for $x = x_1, x_2, \ldots, x_n$ $p''(x_1) = f''(x_1)$, $p''(x_2) = f''(x_2)$; (δ) $a < x_1$, $x_n = b$, $p(x) = f(x)$ for $x = a, x_1, x_2, \ldots, x_{n-1}$, $p' = f'$ for $x = a, x_1, x_2, \ldots, x_n$. Consider the function $g = f-p$, $L(f) = 0$ implies $L(g) = 0$ and this will imply $g(b) = 0$. It will be found that $g''(x)$ vanishes at $2n$ distinct points in $a \leq x \leq b$. Therefore, by successive applications of Rolle's Theorem $g^{(2n+1)}(x)$ vanishes at least once in (a, b). Therefore $f^{2n+1}(x) = 0$ for some value ξ of x in (a, b) and the formula is simplex.

(ii) Let $p(x)$ be a polynomial of degree $2n-2$ such that $p = f$ for $x = x_1, x_2, \ldots, x_{n-1}$, $p' = f'$ for $x = x_1, x_2, \ldots, x_n$. Consider the function $g = f-p$. $L(f) = 0$ implies $L(g) = 0$ which implies $g(b) = 0$. It will be seen that $g'(x)$ vanishes at $2n-1$ distinct points in (a, b) so that $g^{(n-1)}(x)$ *i.e.*, $f^{(2n-1)}(x)$ vanishes at least once in (a, b), *i.e.*, the formula is simplex.

(iii) Let $p(x)$ be a polynomial of degree $2n-1$ at most such that (α) when $x_n < b$, $p = f$ for $x = x_1, x_2, \ldots, x_n$, $p' = f'$ for $x = x_1, x_2, \ldots, x_n$; ($\beta$) when $x_n = b$, $p = f$ for $x = x_1, x_2, \ldots, x_{n-1}$. $p' = f'$ for $x = x_1, x_2, \ldots, x_n$, $p''(x_1) = f''(x_1)$. Consider $g = f-p$; $L(f) = 0$ implies $L(g) = 0$ which again implies $g(b) = 0$, It will be found that $g''(x) = 0$ at $2n-1$ distinct points in (a, b). Therefore, $g^{(2n)}(x)$, *i.e.* $f^{(2n)}(x) = 0$ at least at one point in (a, b). Hence the formula is simplex. Evidently when $b = x_n$ and the order is $2n$, a similar proof will show that the formula is also simplex.

(iv) Obvious.

Thus, each of the following formulae

$$L(f) = f(b)-f(a)$$

$$L(f) = f(b)-f(a)-(b-a)f'\left(\frac{a+b}{2}\right)$$

$$L(f) = f(1)-f(0)-\{5f'(\alpha)+8f'(\tfrac{1}{2})+5f'(\beta)\}/18, \quad (x-\alpha)(x-\beta) = x^2-x+\tfrac{1}{10} = 0$$

$$L(f) = f(3)-f(0)-9\{f(2)-f(1)\}-6\{f'(2)+f'(1)\}$$

which are respectively of orders 1, 3, 7 and 5 are simplex by Theorem 1(*i*). The following five formulae

$$L(f) = f(b)-f(a)-\frac{b-a}{2}\,[f'(a)+f'(b)]$$

$$L(f) = f(b)-f(a)-\frac{b-a}{6}\left\{f'(a)+f'(b)+4f'\left(\frac{a+b}{2}\right)\right\}$$

$$L(f) = 11\{f(3)-f(0)\}+27\{f(2)-f(1)\}-3\{f'(3)+9f'(2)+9f'(1)+f'(0)\}$$

$$L(f) = 25\{f(4)-f(0)\}+160\{f(3)-f(1)\}-6\{f'(4)+f'(0)+16f'(3)+16f'(1)+36f'(2)\}$$

$$\begin{aligned} L(f) = {} & 49\{f(6)-f(0)\}+924\{f(5)-f(1)\}+2625\{f(4)-f(2)\} \\ & -10\{f'(6)+f'(0)+36f'(5)+36f'(1)+225f'(4)+225f'(2)+400f'(3)\} \end{aligned}$$

which are of orders 3, 5, 7, 9, 13 respectively are simplex by Theorem 1(*ii*).

The following three formulae

$$L(f) = f(b)-f(a)-(b-a)f'(a)$$

$$L(f) = f(3)-f(0)-\tfrac{3}{4}\{f'(0)+3f'(2)\}$$

$$L(f) = f(3)-f(0)-\tfrac{3}{4}\{f'(3)+3f'(1)\}$$

which are of orders 2, 4, 4 respectively are simplex by Theorem 1(*iii*).

Lastly the following three formulae

$$L(f) = f(2)-2f(1)+f(0)-\tfrac{1}{2}\{f'(2)-f'(0)\}$$

$$L(f) = 11\{f(4)+f(0)\}+16\{f(3)+f(1)\}-54f(2)-3\{f'(4)+f'(0)+8f'(3)-8f'(1)\}$$

$$\begin{aligned} L(f) = {} & 136\{f(6)+f(0)\}+1488\{f(5)+f(1)\}+375\{f(4)+f(2)\}-400f(3) \\ & -30\{f'(6)-f'(0)+24f'(5)-24f'(1)+75f'(4)-75f'(2)\} \end{aligned}$$

which are of orders 4, 8, 12 respectively are simplex by Theorem 1(*iv*).

Theorem 2. (*a*) *If the formula*

$$L(f) = f(2m)-f(0)-2m\sum_{s=0}^{2m} A_s f'(s)$$

be of order $2m+3$, *then it is simplex.*

(*b*) *If the formula*

$$L(f) = f(2m-1)-f(0)-(2m-1)\sum_{s=0}^{2m-1} A_s f'(s)$$

be of order $2m+1$, *then it is simplex.*

Corollaries: (*i*) *If in* 2(*a*) $A_{2m} = A_0 = 0$ *and the order is* $2m+1$, *the formula is simplex*.

(*ii*) *If in* 2(*b*) $A_{2m-1} = A_0 = 0$, *and the order is* $2m-1$, *the formula is simplex.*

Proof: Let $p(x)$ be a polynomial of degree $2m+1$ at most such that $f' = p'$ for $x = 0, 1, 2, \ldots, 2m$. Consider $g = f-p$ we have

$$f'(x) = p'(x)+x(x-1)(x-2)\ldots(x-2m)f'(x, 0, 1, 2, \ldots, 2m)$$

$$\therefore \int_0^{2m} f'(x)dx = \int_0^{2m} p'(x)dx+\int_0^{2m} x(x-1)\ldots(x-2m)f'(x, 0, 1, 2, \ldots, 2m)dx$$

$$\therefore\quad 0 = L(f) = L(g) = g(2m)-g(0) = \int_0^{2m} g'(x)dx = \int_0^{2m} f'(x)dx - \int_0^{2m} p'(x)dx$$

$$= \int_0^{2m} x(x-1)\ldots(x-2m)f'(x,0,1,2,\ldots,2m)dx = Af^{2m+3}(\xi) \text{ where } A \neq 0.$$

The last result follows easily from the facts: (*i*) $\int_0^{2m} x(x-1)\ldots(x-2m)dx = 0$, (*ii*) $F(x) = \int_0^x t(t-1)\ldots(t-2m)dt$ keeps a constant sign for $0 \leq x \leq 2m$ and (*iii*) the first mean value theorem of the integral calculus. Hence the formula is simplex.

Similarly, we can prove Theorem 2(*b*) and the corollaries. (*Cf.* Milne Thomson (1933), Articles 7.13 and 7.14).

From the above theorem the fact that all the usual formulæ of the open and closed types are simplex, will follow. In particular all the ten formulae in Group I of Bickley's paper are simplex. A similar conclusion will follow for each of the following formulæ of the open type

$$L(f) = f(3)-f(0)-\tfrac{3}{2}[f'(1)+f'(2)]$$
$$L(f) = f(5)-f(0)-\tfrac{5}{24}[11f'(1)+f'(2)+f'(3)+11f'(4)]$$
$$L(f) = f(6)-f(0)-\tfrac{3}{10}[11f'(1)-14f'(2)+26f'(3)-14f'(4)+11f'(5)]$$
$$L(f) = f(7)-f(0)-\tfrac{7}{1440}[611f'(1)-453f'(2)+562f'(3)+562f'(4)-453f'(5)+611f'(6)]$$

which are respectively of orders 3, 5, 7 and 7.

Theorem 3. *If the formula*

$$L(f) = f(b)-f(a)-[B_1f'(a)+B_2f'(b)+B_3f'(x_1)+B_4f'(x_2)+\cdots+B_{n+2}f'(x_n)]$$

where in the interval (a, b) *the product* $(x-x_1)(x-x_2)\ldots(x-x_n)$ *has no zero, be of order* $n+3$, *then it is simplex. If further either* B_1 *or* B_2 *is zero and the order is* $n+2$, *the formula will be simplex. Lastly if* $B_1 = B_2 = 0$ *and the order is* $n+1$, *the formula is also simplex.*

Proof: As before

$$L(f) = \int_a^b (x-a)(x-b)(x-x_1)(x-x_2)\ldots(x-x_n)f'(x,a,x_1,x_2,\ldots,x_n,b)dx$$

The quantity $(x-a)(x-b)(x-x_1)\ldots(x-x_n)$ keeps a constant sign in (a, b). Therefore, by the 1st Mean Value Theorem of the Integral calculus

$$L(f) = \frac{f^{(n+3)}(\xi)}{(n+3)!}\int_a^b (x-a)(x-b)(x-x_1)(x-x_2)\ldots(x-x_n)dx$$

$$= A.f^{(n+3)}(\xi), \quad \text{where } A \neq 0.$$

Therefore $L(f) = 0$ implies $f^{(n+3)}(\xi) = 0$ and the formula is simplex. The remaining part of the theorem will follow by slight modification of the above proof.

ILLUSTRATIONS

The following five formulæ

(i) $L(f) = f(1)-f(0)-[5f'(0)+8f'(1)-f'(2)]/12$

(ii) $L(f) = f(1)-f(0)-[9f'(0)+19f'(1)-5f'(2)+f'(3)]/24$

(iii) $L(f) = f(1)-f(0)-[251f'(0)+646f'(1)-264f'(2)+106f'(3)-19f'(4)]/720$

(iv) $L(f) = f(1)-f(0)-[55f'(0)-59f'(-1)+37f'(-2)-9f'(-3)]/24$

(v) $L(f) = f(1)-f(0)-\frac{1}{2}[5f'(2)-3f'(3)]$

which are respectively of orders 4, 5, 6, 5 and 3 are simplex by the above theorem.

Theorem 4. *If the formula*

$$L(f) = f(2m)-f(0)-2m\sum_{s=1}^{m} A_s f'(2s-1)$$

where m is odd, be of order $m+2$, then it is simplex.

Proof: Evidently

$$L(f) = \int_0^{2m} (x-1)(x-3)\ldots(x-2m+1)f'(x, 1, 3, 5, \ldots, \overline{2m-1})dx$$

Integrating by parts and remembering that $F_m(0) = F_m(2m) = 0$ where

$$F_m(x) = \int_0^x (t-1)(t-3)\ldots(t-2m+1)dt,$$

we get

$$L(f) = -\int_0^{2m} F_m(x)f'(x, x, 1, 3, 5, \ldots, \overline{2m-1})dx = -Af^{(m+2)}(\xi)$$

where $A<0$ and $0\leq\xi\leq 2m$. The above result follows from the 1st Mean Value Theorem of the Integral calculus which is applicable here since $F_m(x)<0$ for $0<x<2m$. The last result can be established in the following manner. Since $F_m(x) = F_m(2m-x)$, it is enough if we can show that $F_m(x)<0$ for $0<x<m$. Now consider

$$F_m(x) = \left(\int_0^1+\int_1^3+\int_3^5+\cdots\right)(t-1)(t-3)\ldots(t-2m+1)dt, \quad (0\leq x<m)$$

Here the first integral is negative and from the second integral onwards the terms are alternately positive and negative, each term being greater in absolute value than the term succeeding. Besides $F_1(x) = [(x-1)^2-1]/2<0$ for $0<x<2$ and $F_3(x) = x(x-6)\times[(x-3)^2+1]/4<0$ for $0<x<6$.

$$\therefore \int_0^3 (t-1)(t-3)(t-5)\ldots(t-2m+1)dt = \Big[F_3(t)(t-7)(t-9)\ldots(t-2m+1)\Big]_0^3 - \int_0^3 F_3(t)\frac{d}{dt}\{(t-7)\ldots(t-2m)\}dt$$

Therefore the 1st integral in $F_m(x)$ determines the sign of $F_m(x)$ and is negative.

ILLUSTRATIONS

The formulæ

$$L(f) = f(6)-f(0)-3[3f'(1)+2f'(3)+3f'(5)]/4$$

$$L(f) = f(10)-f(0)-10[275f'(1)+100f'(3)+402f'(5)+100f'(7)+275f'(9)]/1152$$

which are respectively of orders 5 and 7, are simplex by the above theorem.

4. Often we come across mixed formulæ *i.e.* formulae obtained by combining two or more formulae of the simplex type. Sometimes these combined formulae are also simplex. Consider, for example,

$$\begin{aligned} L(f) &= f(4)-f(0)-\tfrac{1}{3}\{f'(0)+f'(4)+4f'(1)+4f'(3)+2f'(2)\} \\ &= f(4)-f(2)-\tfrac{1}{3}\{f'(4)+4f'(3)+f'(2)\}+f(2)-f(0)-\tfrac{1}{3}\{f'(2)+4f'(1)+f'(0)\} \\ &= -[f^{(5)}(\xi_1)+f^{(5)}(\xi_2)]/90 = f^{(5)}(\xi)/45. \end{aligned}$$

Or, consider the formula

$$\begin{aligned} L(f) &= f(4)-f(0)-2[f'(1)+f'(3)]. \\ &= f(4)-f(2)-2f'(3)+f(2)-f(0)-2f'(1) \\ &= \tfrac{1}{3}f'''(\xi_1)+\tfrac{1}{3}f'''(\xi_2) = \tfrac{2}{3}f'''(\xi) \end{aligned}$$

DEPARTMENT OF MATHEMATICS & STATISTICS,
DACCA UNIVERSITY

References

Bickley, W. G. (1939), *Mathematical Gazette*, **23**, 352-59.
Daniel, P. J., (1940), *Mathematical Gazette*, **24**, 238-44.
Milne-Thomson, L. M., (1933), *The Calculus of Finite Differences*, 1st ed (McMillan).

HOMOTOPY GROUPS OF SOME MAPPING SPACES OF SPHERES

BY

SZE-TSEN HU

(*Communicated by the Secretary—Received October 10, 1947*)

1. INTRODUCTION.

Let Y be a connected, locally contractible, separable metric space. The fundamental group of the mapping space $Y^{S^m}(x_0, y_0)$, *i.e.* the metric space consisting of all the mappings $f(S^m) \subset Y$ such that $f(x_0) = y_0$ where $x_0 \in S^m$, $y_0 \in Y$ are given points, is defined by W. Hurewicz (1935), as the $(m+1)$-th homotopy group of Y. The fundamental group of the null-component of the mapping space Y^{S^m} has been discussed by M. Abe (1939). The structures of higher homotopy groups of the both spaces are determined in terms of the homotopy groups of Y by the author in a recent work (1947). The present paper is to give two generalizations including all these known results. Throughout this paper, a mapping is a continuous transformation.

Let Y_0 be a connected and locally contractible subset of Y, and y_0 be a given point of Y_0. Let S^m be an m-sphere and x_0, x_1 be two distinct fixed points of S^m. Let us denote by Λ^m the metric space which consists of all the mappings $f(S^m) \subset Y$ such that $f(x_1) \in Y_0$, and by Ω^m the metric space which consists of all the mappings $f(S^m) \subset Y$ such that $f(x_0) = y_0$, $f(x_1) \in Y_0$. Let Λ^m_*, Ω^m_* denote respectively the components of Λ^m, Ω^m which contain the constant mapping $O(S^m) = y_0$. Our main results are embodied in the following theorem.

Theorem: *The fundamental group* $\pi^1 = \pi^1(\Lambda^m_*)[\pi^1(\Omega^m_*)]$ *has an invariant subgroup* π *which is isomorphic with* $\pi^{m+1}(Y)$. *The quotient group* π^1/π *is isomorphic with* $\pi^1(Y_0)[\pi^2(Y, Y_0)]$. *The r-th homotopy group* $\pi^r = \pi^r(\Lambda^m_*)[\pi^r(\Omega^m_*)]$, $r > 1$, *is isomorphic with the direct sum* $\pi^{m+r}(Y) + \pi^r(Y_0)[\pi^{m+r}(Y) + \pi^{r+1}(Y, Y_0)]$.

By $\pi^n(Y, Y_0)$ we denote the n-th relative homotopy group of Y in relation to Y_0, (Hurewicz and Steenrod, 1941).

2. The group $\lambda_r{}^n(Y, Y_0)$ and the group $\omega_r{}^n(Y, Y_0)$.

Let us consider, in the Euclidean $(n+1)$-space, the $(n+1)$-element E^{n+1} defined by the inequality

$$\xi_0{}^2 + \xi_1{}^2 + \cdots + \xi_n{}^2 \leq 1.$$

Let S^r, $r \leq n$, be the r-sphere defined by

$$\xi_0{}^2 + \xi_1{}^2 + \cdots + \xi_r{}^2 = 1\ ; \quad \xi_i = 0,\ i > r.$$

Let $E_1{}^r$, $E_2{}^r$ denote the subsets of S^r, $r \leq n$, defined by $\xi_0 \geq 0$, $\xi_0 \leq 0$; and $D_1{}^r$, $D_2{}^r$, the subsets of S^r defined by $\xi_r \geq 0$, $\xi_r \leq 0$. Let p_0 denote the point $(0, -1, 0, \ldots, 0)$ of S^n, then $p_0 \in S^r$ for each $r \geq 1$.

We shall consider the totality of the mappings $f(S^n) \subset Y$ such that $f(D_2^r) \subset Y_0$ and $f(S^{r-1}) = y_0$ $[f(S^r) \subset Y_0$ and $f(p_0) = y_0]$. These mappings can be divided into homotopy classes relative to $(D_2^r, Y_0; S^{r-1}, y_0)$[to $(S^r, Y_0; p_0, y_0)$]. We shall denote the class of the mapping f by $\tilde{f}$. We denote by O the constant mapping $O(S^n) = y_0$.

Let f_1 and f_2 be any two of these mappings. It can be easily proved that there exist mappings $f_i'(S^n) \subset Y$, $(i = 1, 2)$, such that

$$(1) \quad f_i'(E_j^n) = y_0, \quad (i, j = 1, 2;\ i \neq j),$$

$$(2) \quad f_i' \simeq f_i \text{ rel. } (D_2^r, Y_0; S^{r-1}, y_0)[\text{rel. } (S^r, Y_0; p_0, y_0)].$$

We define a mapping $f(S^n) \subset Y$ by taking

$$f = f_i' \text{ on } E_i^n (i = 1, 2),$$

then f is also a mapping of the kind we are considering. It can be verified that the class $\tilde{f}$ depends only on the classes $\tilde{f}_1$ and $\tilde{f}_2$. We define

$$\tilde{f} = \tilde{f}_1 + \tilde{f}_2;$$

it can be easily seen that the operation + has all the properties of a group operation. The zero element is the class $\tilde{O}$; and the negative element of $\tilde{f}$ is the class $\widetilde{f\theta}$, where $\theta(S^n) \subset S^n$ is the mapping which maps each point $x \in S^n$ on its mirror image $\theta(x)$ in the hyperplane $\xi_0 = 0$.

The group obtained in this way will be denoted by $\lambda_r^n = \lambda_r^n(Y, Y_0)[\omega_r^n = \omega_r^n(Y, Y_0)]$. When $Y_0 = Y$, it reduces to the group $k_{r-1}^n(Y)[\pi^n(Y)]$; when $Y_0 = y_0$, it reduces to the $\pi^n(Y)[k_r^n(Y)]$, *cf.* (Hu, 1947, §4). It should be remarked that although we use the additive notation, the group $\lambda_r^n[\omega_r^n]$ is not necessarily commutative when $r = 1$.

The following theorem can be proved with the same argument used in (Hu, 1947), §5.

Theorem 2. 1. *For each $r \geq 1$, we have the isomorphisms*

$$\pi^r(\Lambda_*^m) \approx \lambda_r^{m+r}(Y, Y_0),$$

$$\pi^r(\Omega_*^m) \approx \omega_r^{m+r}(Y, Y_0).$$

Corollary 2. 2. *The groups λ_r^n and ω_r^n are commutative if $r > 1$.*

3. The principal and inessential subgroups.

A spherical mapping $f(S^n) \subset Y$ is said to be principal, in case $f(D_2^n) = y_0$. An element of $\lambda_r^n[\omega_r^n]$ is said to be principal, if it can be represented by a principal mapping. The following theorem can be proved by the argument used in (Hu, 1947, §6).

Theorem 3.1. *The principal elements of $\lambda_r^n(Y, Y_0)[\omega_r^n(Y, Y_0)]$ form a subgroup $\mu_r^n(Y, Y_0)[\rho_r^n(Y, Y_0)]$, called the principal subgroup, which is isomorphic with $\pi^n(Y)$.*

A mapping $f(S^n) \subset Y$ is said to be inessential, if it has an extension $f^*(E^{n+1}) \subset Y$. An element of $\lambda_r^n[\omega_r^n]$ is said to be inessential, if it can be represented by an inessential mapping. It is obvious that all representatives of an inessential element are inessential.

Theorem 3.2. *The inessential elements of* $\lambda_r^n(Y, Y_0)[\omega_r^n(Y, Y_0)]$ *form a subgroup* $\nu_r^n(Y, Y_0)[\theta_r^n(Y, Y_0)]$, *called the inessential subgroup, which is isomorphic with* $\pi^r(Y_0)[\pi^{r+1}(Y, Y_0)]$.

The first part of the theorem is trivial, the second part will be proved in §4. The following theorem is also evident.

Theorem 3.3. *The principal subgroup* $\mu_r^n[\rho_r^n]$ *and the inessential subgroup* $\nu_r^n[\theta_r^n]$ *of the group* $\lambda_r^n[\omega_r^n]$ *have the zero element as their only common element.*

4. Natural homomorphisms and the structures.

Let α be an arbitrary element of $\lambda_r^n(Y, Y_0)[\omega_r^n(Y, Y_0)]$, and $f(\dot{S}^n)\subset Y$ be a representative of α. Let ϕ be the partial mapping defined by $\phi = f\,|\,D_2^r[\phi = f\,|\,D_2^{r+1}]$; then ϕ represents and element β of $\pi^r(Y_0)[\pi^{r+1}(Y, Y_0)$ with $p_0\in D_2^{r+1}$ as base point]. This transformation $\alpha\rightarrow\beta$ can be easily seen to be a homomorphism h of the group $\lambda_r^n[\omega_r^n]$ onto the group $\pi^r(Y, Y_0)[\pi^{r+1}]$, which will be called the natural homomorphism $\boldsymbol{h}$.

Theorem 4.1. *The kernel of the natural homomorphism* h *of* $\lambda_r^n[\omega_r^n]$ *onto* $\pi^r(Y_0)[\pi^{r+1}(Y, Y_0)]$ *is the principal subgroup* $\mu_r^n[\rho_r^n]$.

Proof. If $\alpha\in\mu_r^n[\rho_r^n]$, it is evident that $h(\alpha) = 0$. Conversely, if $h(\alpha) = 0$ and f represents α, then we have

$$\phi\simeq 0 \text{ rel. } (S^{r-1};\, y_0) \text{ in } Y_0[\text{rel. } (S^r, Y_0;\, p_0, y_0) \text{ in } Y].$$

since $\phi = f\,|\,D_2^r[\phi = f\,|\,D_2^{r+1}]$, there exists a mapping $f'(S^n)\subset Y$ such that

(1) $f'\simeq f$ rel. $(D_2^r, Y_0;\, S^{r-1}, y_0)[(S^r, Y_0;\, p_0, y_0)]$,

(2) $f'(D_2^r) = y_0[f'(D_2^{r+1}) = y_0]$.

It is clear that $D_2^r[D_2^{r+1}]$ is a retract by deformation of D_2^n relative to itself; therefore, there exists a mapping $f''(S^n)\subset Y$ such that

(1) $f''\simeq f'$ rel. $D_2^r[D_2^{r+1}]$,

(2) $f''(D_2^n) = y_0$.

Hence it follows that α is a principal element, and the theorem has been proved.

Corollary 4.2. *The principal subgroup* $\mu_r^n[\rho_r^n]$ *of the group* $\lambda_r^n[\omega_r^n]$ *is invariant.*

Theorem 4.3. *The natural homomorphism* h *maps the inessential subgroup* $\nu_r^n[\theta_r^n]$ *of the group* $\lambda_r^n[\omega_r^n]$ *isomorphically onto the group* $\pi^r(Y_0)[\pi^{r+1}(Y, Y_0)]$.

Proof. Let β be an arbitrary element of $\pi^r(Y_0)[\pi^{r+1}(Y, Y_0)]$, and let $\phi(D_2^r)\subset Y_0$, $\phi(S^{r-1}) = y_0[\phi(D_2^{r+1})\subset Y,\ \phi(S^r)\subset Y_0,\ \phi(p_0) = y_0]$ be a representative of β. Since $D_2^r[D_2^{r+1}]$ is a retract of E^{n+1}, ϕ has an extension $f^*(E^{n+1})\subset Y$. Let f be the partial mapping defined by $f = f^*\,|\,S^n$ and let α be the element of $\lambda_r^n[\omega_r^n]$ represented by f, then α is inessential and $h(\alpha) = \beta$. Hence h maps $\nu_r^n[\theta_r^n]$ onto $\pi^r(Y_0)[\pi^{r+1}(Y, Y_0)]$.

Next let $\alpha\in\nu_r^n[\theta_r^n]$ such that $h(\alpha) = 0$, then $\alpha\in\mu_r^n[\rho_r^n]$. It follows from Theorem 3.3 that $\alpha = 0$, and the theorem is proved.

Now let us consider the group λ_r^n, $r>1$, $[\omega_r^n, r\geq 1]$. Let α be an arbitrary element of $\lambda_r^n[\omega_r^n]$ and $f(S^n)\subset Y$ be a representative of α. In the case we consider, $f(p_0)=y_0$, therefore, f represents an element $\beta\in\pi^n(Y)$. It can be easily seen that the transformation $\alpha\rightarrow\beta$ is a homomorphism k of the group $\lambda_r^n[\omega_r^n]$ onto the group $\pi^n(Y)$ with p_0 as base point, which will be called the natural homomorphism k.

The following two theorems are trivial.

Theorem 4.4. *The kernel of the natural homomorphism k of λ_r^n, $r>1$, $[\omega_r^n, r\geq 1]$, onto $\pi^n(Y)$ is the inessential subgroup $\nu_r^n[\theta_r^n]$.*

Corollary 4.5. *The inessential subgroup $\nu_r^n[\theta_r^n]$ of the group λ_r^n, $r>1$, $[\omega_r^n, r\geq 1]$, is invariant.*

Theorem 4.6. *The natural homomorphism k maps the principal subgroup $\mu_r^n[\rho_r^n]$ isomorphically onto $\pi^n(Y)$.*

The following theorems can be proved by easy group-theoretic arguments.

Theorem 4.7. *Each element α of $\lambda_r^n[\omega_r^n]$ can be expressed uniquely in the forms*

$$\alpha=\beta+\gamma=\delta+\beta,$$

where β is inessential and γ, δ are principal.

Theorem 4.8.

$$\lambda_r^n=\mu_r^n+\nu_r^n,\quad r\geq 1;$$

$$\omega_r^n=\rho_r^n+\theta_r^n,\quad r>1;$$

$$\omega_1^n=\rho_1^n\times\theta_1^n.$$

Theorem 4.9. $\lambda_1^n=\mu_1^n\times\nu_1^n$ *if and only if* ν_1^n *is invariant.*

The theorem mentioned in the introduction is an immediate consequence of the theorems in §3 and §4.

MANCHESTER UNIVERSITY.

References

Abe, M., (1939), *Japanese Journal of Mathematics*, 16, 169-176.

Hu, S. T., (1947), *Annals of Mathematics*, 48, 717-734.

Hurewicz, W., (1935), *Proc, Akad. Amsterdam*, 38, 112-119.

Hurewicz, W. and Steenrod, N. E., (1941), *Proc. Nat. Acad. Sci* USA, 37, 60-64

A NOTE ON THE EQUILIBRIUM OF FLUID MATTER IN A STEADY DIFFERENTIAL ROTATION*

By

N. L. Ghosh

(*Received December 22, 1947*)

Introduction

The problem of steady differential rotation of a mass of frictionless fluid about an axis of symmetry has received a good deal of attention from various investigators. Pierre Dive (1930) in course of his work on the subject has taken notice of the case in which the gravitational potential becomes purely a function of the density of the fluid. In course of investigations into the problem the present author noticed that the assumption of the potential being a function of the density alone imposes certain restrictions on the density distribution of the fluid mass. The present paper contains a discussion of this point. It is shown that when the density ϱ is a function of the gravitational potential only $\nabla^2\varrho$ and $|\nabla\varrho|^2$ must both be functions of ϱ only. As illustration two different configurations of equilibrium under the conditions envisaged have been studied. The gravitational potential, in each case, has been assumed to be produced by a very heavy central spherical core, the gravitational effect of the fluid itself being neglected as small. No pressure-density relation has been assumed. It has been found that the fluid, under the circumstances mentioned may remain in equilibrium-configuration as a spheroid with differential rotation round the heavy core, very much like a rotating thin atmosphere surrounding a heavy spherical nucleus, or even as an anchor-ring (within which the material is in rotation) concentric with the core. A peculiar feature of both the solutions is that the density, which is everywhere small, increases while the angular velocity diminishes outwards.

1

The equations of motion of a mass of perfect fluid rotating steadily about an axis of symmetry are given, in cylindrical co-ordinates, by

$$\omega^2 r = -\frac{\partial\Phi}{\partial r} + \frac{1}{\varrho}\frac{\partial p}{\partial r} \tag{1}$$

$$0 = -\frac{\partial\Phi}{\partial z} + \frac{1}{\varrho}\frac{\partial p}{\partial z} \tag{2}$$

$$\nabla^2\Phi = -4\pi\varrho. \tag{3}$$

From (1) and (2) we have

$$dp = \varrho d\Phi + \varrho\omega^2 r dr. \tag{4}$$

---

* The results of this paper were communicated and discussed at a session of the Indian Science Congress.

Eqn. (4) shows that if Φ is a function of ϱ alone, since dp and $\varrho d\Phi$ are exact differentials, $\varrho\omega^2 r dr$ must also be so and hence $\varrho\omega^2$ must be a function of 'r' alone. Thus we arrive at the first conclusion that "if Φ is a function of ϱ, $\varrho\omega^2$ must be a function of r" and conversely "if $\varrho\omega^2$ be a constant or a function of r alone Φ must be a function of ϱ."

Now let

$$\varrho d\Phi = dF(\varrho) \tag{5}$$

so that

$$\frac{\partial\Phi}{\partial r} = \frac{1}{\varrho}\frac{\partial F}{\partial r} \tag{6}$$

and

$$\frac{\partial\Phi}{\partial z} = \frac{1}{\varrho}\frac{\partial F}{\partial z} \tag{7}$$

Now, Eqn. (3) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{\partial^2\Phi}{\partial z^2} = -4\pi\varrho.$$

Hence substituting from (6) and (7) we obtain

$$\frac{1}{\ }\frac{\partial}{\partial r}\left(\frac{r}{\varrho}\frac{\partial F}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\varrho}\frac{\partial F}{\partial z}\right) = -4\pi\varrho \tag{8}$$

where F is a function of ϱ alone.

Equation (8) can be put in the form

$$\begin{aligned} -4\pi\varrho &= \frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{r}{\varrho}\frac{\partial F}{d\varrho}.\frac{\partial\varrho}{\partial r}\right\} + \frac{\partial}{\partial z}\left\{\frac{1}{\varrho}\frac{dF}{d\varrho}.\frac{\partial\varrho}{\partial z}\right\} \\ &= F'(\varrho).\nabla^2(\log\varrho) + F''(\varrho)\left\{\frac{1}{\varrho}\left(\frac{\partial\varrho}{\partial r}\right)^2 + \frac{1}{\varrho}\left(\frac{\partial\varrho}{\partial z}\right)^2\right\} \\ &= F'(\varrho).\nabla^2(\log\varrho) + F''(\varrho).\frac{|\nabla\varrho|^2}{\varrho} \end{aligned}$$

so that

$$F''(\varrho).|\nabla\varrho|^2 + F'(\varrho).\varrho\nabla^2(\log\varrho) + 4\pi\varrho^2 = 0. \tag{9}$$

In any possible case, equation (9) must be integrable; hence $|\nabla\varrho|^2$ and $\nabla^2(\log\varrho)$ must both be functions of ϱ. But,

$$\nabla^2(\log\varrho) = \frac{1}{\varrho}\nabla^2\varrho - \frac{1}{\varrho^2}|\nabla\varrho|^2$$

Hence we conclude that if the gravitational potential be a function of the mass-density $|\nabla\varrho|^2$ and $\nabla^2\varrho$ should both be functions of ϱ. The physical interpretation of $|\nabla\varrho| = f(\varrho)$ is as follows. Let us start from a level surface $\varrho =$ const. Then the neighbouring level surface can be constructed from it by displacing every element of this surface normal to itself through the same infinitesimal distance. This fixes the geometrical character of the level surfaces. It may be noted that the simplest surface of this type is a sphere.

2

Case I. We shall now consider the particular case of the problem in which the potential is due entirely to a central sphere of a very heavy material, so that the self-gravitation of the fluid at any point of itself may be considered, by comparison with the field of the central sphere, negligibly small.

The equations of the problem are

$$\omega^2 r = -\frac{\partial \Phi}{\partial r} + \frac{1}{\varrho}\frac{\partial p}{\partial r}$$

$$0 = -\frac{\partial \Phi}{\partial z} + \frac{1}{\varrho}\frac{\partial p}{\partial z}$$

$$\Phi = \frac{GM}{R} = \frac{GM}{\sqrt{r^2+z^2}} \tag{10}$$

As before, we notice that if $\varrho\omega^2$ be a constant or a function of r alone, ϱ must be a function of Φ. To have a suitable case satisfying the above conditions we assume

$$\omega^2 = \frac{\omega_0^{\,2} a^3}{(r^2+z^2)^{3/2}} \tag{11}$$

and

$$\varrho = \frac{\varrho_0}{a^3}\cdot(r^2+z^2)^{3/2} \tag{12}$$

where a is the radius of the central sphere, and ω_0, ϱ_0 are the values of ω and ϱ at the surface of the sphere. Hence,

$$\varrho\omega^2 = \varrho_0\omega_0^{\,2} \tag{13}$$

and therefore a constant. Also

$$\varrho = \frac{\varrho_0(r^2+z^2)^{3/2}}{a^3} \quad \text{and} \quad \Phi = \frac{GM}{(r^2+z^2)^{1/2}}$$

$$\therefore \quad \varrho = \frac{\varrho_0}{a^3}\left(\frac{GM}{\Phi}\right)^3. \tag{14}$$

Equations (13) and (14) ensure both the conditions for integrability. Now from equations (1) and (2) we have

$$p = \int \varrho\, d\Phi + \int \varrho\omega^2 r\, dr + C,$$

where C is an absolute constant. Substituting (14) and integrating we have,

$$p = \frac{\varrho_0}{2}\left[-\frac{GM}{a^3}(r^2+z^2) + \omega_0^{\,2} r^2\right] + C, \tag{15}$$

If $p = p_0$, where $r = 0$, $z = a$,

$$C = p_0 + \frac{\varrho_0 GM}{2a} = \frac{\varrho_0}{2}\left[\frac{2p_0}{\varrho_0} + \frac{GM}{a}\right]. \tag{16}$$

Hence substituting in (15), we have

$$p = \frac{\varrho_0}{2}\left[\left(\frac{2p_0}{\varrho_0} + \frac{GM}{a}\right) - \left(\frac{GM}{a^3} - \omega_0^2\right)r^2 - \frac{GM}{a^3}\cdot z^2\right]. \tag{17}$$

OBSERVATIONS

(*i*). Equation (17) shows that the spheroidal form is possible if

$$\frac{GM}{a^3} - \omega_0^2 \geq 0$$

or

$$\omega_0^2 < \frac{GM}{a^3}$$

i.e.,

$$\omega_0^2 < \frac{g}{a} \tag{18}$$

where g is the attraction at the surface of the sphere. But ω_0 is the angular velocity at the surface of the sphere. Hence, the spheroidal form is possible if

$$a\omega_0^2 < g$$

or the centrifugal force at the equator is less than the attraction there.

(*ii*). Again, it may be noted, that if $(GM/a^3 - \omega_0^2)$ is positive,

$$\frac{GM}{a^3} - \omega_0^2 < \frac{GM}{a^3}.$$

But the equatorial and polar semi-axes are respectively

$$(2p_0/\varrho_0 + GM/a)^{1/2} \Big/ (GM/a^3 - \omega_0^2)^{1/2}$$

and

$$(2p_0/\varrho_0 + GM/a)^{1/2} \Big/ (GM/a^3)^{1/2}.$$

Hence, the spheroid is oblate.

(*iii*). Besides, the equatorial extension is inversely porportional to the excess of GM/a^3 over ω_0^2 that is to $(g - a\omega_0^2)/a$, or to the excess of the attraction at the surface over the centrifugal force.

(*iv*). If α and β denote the equatorial and polar radii we have

$$\alpha = (2p_0/\varrho_0 + GM/a)^{1/2} \Big/ (GM/a^3 - \omega_0^2)^{1/2}\,; \quad \beta = (2p_0/\varrho_0 + GM/a)^{1/2} \Big/ (GM/a^3)^{1/2} \tag{19}$$

The ratio of the axes is

$$\frac{\alpha}{\beta} = (GM/a^3)^{1/2} \Big/ (GM/a^3 - \omega_0^2)^{1/2} = \sqrt{g}/\sqrt{(g - a\omega_0^2)}. \tag{20}$$

The eccentricity

$$e = \omega_0\sqrt{(a/g)}. \tag{21}$$

(*v*). The total mass m of the atmosphere satisfies $m_1 < m < m_2$, where

$$m_1 = \int_0^\beta 4\pi R^2.\varrho dR \quad \text{and} \quad m_2 = \int_0^\alpha 4\pi R^2.\varrho dR.$$

On evaluation we find

$$m_1 = \tfrac{2}{3}\pi\varrho_0.a^3\left\{\left(\frac{\beta}{a}\right)^6-1\right\} \quad \text{and} \quad m_2 = \tfrac{2}{3}\pi\varrho_0.a^3\left\{\left(\frac{\alpha}{a}\right)^6-1\right\} \tag{22}$$

(*vi*). The results (v) show that, the total mass of the atmosphere may be made arbitrarily small by diminishing ϱ_0 sufficiently. That is, the neglect of the self-gravitation of the fluid may be justified if the density at the surface of the central core be sufficiently small.

(*vii*). The total volume of the atmosphere

$$= \tfrac{4}{3}\pi\beta\alpha^2-\tfrac{4}{3}\pi a^3 = \tfrac{4}{3}\pi a^3\left\{\frac{(2p_0/a\varrho_0+g)^{3/2}}{\sqrt{g}.(g-a\omega_0^2)}-1\right\}. \tag{23}$$

3

Case II. Equilibrium in the form of an anchor-ring.

As before, let us consider a mass of frictionless fluid rotating steadily about an axis of symmetry. The system is supposed to have a heavy spherical attracting nucleus situated with the centre at origin and so heavy that the gravitation at any point of the fluid may be supposed to be due solely to the central body alone.

The equations of motion of any particle of the fluid, in cylindrical co-ordinates are

$$\omega^2 r = -\frac{\partial\Phi}{\partial r}+\frac{1}{\varrho}\frac{\partial p}{\partial r} \tag{1}$$

$$0 = -\frac{\partial\Phi}{\partial z}+\frac{1}{\varrho}\frac{\partial p}{\partial z} \tag{2}$$

$$\Phi = \frac{GM}{(r^2+z^2)^{1/2}} \tag{10}$$

as in case I.

As a trial solution we put

$$p = \frac{p_0}{a^2}\{a^2-z^2-(r-c)^2\} \tag{24}$$

so that

$$p = p_0 \text{ at } r = c, \quad z = 0. \tag{25}$$

Therefore,

$$\left.\begin{aligned}\frac{\partial p}{\partial r} &= -\frac{2p_0}{a^2}(r-c)\\ \frac{\partial p}{\partial z} &= -\frac{2p_0.z}{a^2}\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned}\frac{\partial\Phi}{\partial r} &= -\frac{GM}{R^2}.\frac{r}{R}\\ \frac{\partial\Phi}{\partial z} &= -\frac{GM}{R^2}.\frac{z}{R}\end{aligned}\right\} \quad \text{where} \quad R^2 = r^2+z^2.$$

Substituting these values in the equations of motion and evaluating we have

$$\varrho = \frac{2p_0}{a^2.GM}.R^3 = \frac{\varrho_0 R^3}{c^3}, \text{ (say)} \tag{26}$$

$$\omega^2 = \frac{GM.c}{rR^3} = \frac{\omega_0{}^2 c^4}{rR^3}, \text{ (say)} \tag{27}$$

Equations (26) and (27) show that

$$(i) \quad \varrho\omega^2 = \frac{\varrho_0 \omega_0{}^2 c}{r} \tag{28}$$

and hence $\varrho\omega^2$ is a function of r alone.

$$(ii) \quad \varrho = \frac{\varrho_0 R^3}{c^3}; \quad \Phi = \frac{GM}{R},$$

and hence ϱ is a function of Φ alone or vice-versa. Thus, assumption (24) satisfies the necessary conditions of the problem, with the density and the angular velocity determined by equations (26) and (27) respectively. Hence we may conclude that the values of Φ, p, ϱ, ω given by equations (10), (24), (26) and (27) respectively furnish us with a solution of the problem considered. We shall now proceed to deduce the physical consequences of the solution.

(i). *Pressure*: The equation (24) indicates that the fluid is in equilibrium in the form of an anchor-ring, the centre of its meridian section being at a distance c from the centre of the attracting sphere and its radius being a. The surfaces of equal pressure are all rings of the same nature with merely the radius of the meridian section altering. The pressure, which is necessarily positive everywhere inside, vanishes at the outer surface.

At the centre C of the meridian section the pressure is given by

$$p = p_0.$$

Again

$$\frac{\partial p}{\partial r} = 0 \text{ at } r = c; \qquad \frac{\partial p}{\partial z} = 0 \text{ at } z = 0$$

Further from

$$\frac{\partial^2 p}{\partial r^2} = -\frac{2p_0}{a^2}; \quad \frac{\partial p^2}{\partial z^2} = -\frac{2p_0}{a^2}; \quad \frac{\partial^2 p}{\partial r \partial z} = 0$$

we conclude that the pressure is *maximum* at the point.

(ii). *Density*: From equation (26) we have

$$\varrho = \frac{2p_0.R^3}{a^2.GM} = \frac{\varrho_0 R^3}{c^3}, \quad R \text{ being } \sqrt{(r^2+z^2)}$$

$$\therefore \quad \varrho = \varrho_0 \text{ at } r = c, \quad z = 0.$$

Hence, the density of the fluid at the centre of the circular section is ϱ_0.

Or,

$$\varrho = \varrho_0 \tag{29}$$

at the point C. Now,

$$\varrho_0 = \frac{2p_0.c^3}{a^2GM} = \frac{2p_0.c}{a^2}.\frac{1}{GM/c^2} = \frac{2p_0.c}{a^2.g_0} \tag{30}$$

where, g_0 = the attraction at C.

Equation (26) shows that the density *increases outwards*, the surfaces of equal density being parts of a sphere. A and B being the points nearest to and farthest from the centre of attraction, the rarest parts of the fluid collect at A and the densest at B. The density changes monotonically and hence possesses neither a maximum nor a minimum. Further

$$\frac{\varrho_A}{\varrho_B} = \frac{(\varrho_0/c_3)(c-a)^3}{(\varrho_0/c_3)(c+a)^3} = \left(\frac{c-a}{c+a}\right)^3; \quad \therefore \quad \frac{c}{a} = \frac{\sqrt[3]{\varrho_B}+\sqrt[3]{\varrho_A}}{\sqrt[3]{\varrho_B}-\sqrt[3]{\varrho_A}}.$$

Hence, the ratio of the axial distance of the core of the ring to the radius of its meridian section is determined entirely by the ratio of the highest and lowest densities of the fluid—a rather remarkable result.

(*iii*). *Angular velocity*: From equation (27) we have,

$$\omega^2 = \frac{GMc}{rR^3} = \frac{\omega_0^2.c^4}{rR^3}$$

At C, $r = c$, $z = 0$, $R = r = c$. Therefore,

$$\omega^2 = \omega_0^2,$$

That is, ω_0 in the equation, represents the angular velocity at the centre of the meridian section.

Equation (27) shows obviously that ω diminishes outwards along the equatorial radius, being the greatest at A and the least at B. Now

$$\omega_0^2 = \frac{GM}{c^3} = \frac{1}{c}.\frac{GM}{c^2} = \frac{g_0}{c}$$

where g_0 is the attraction at C, the centre of the meridian section. Thus,

$$\omega_0^2 = \frac{g_0}{c} \tag{31}$$

That is

$$\omega_0^2.c = g_0$$

Hence, the centrifugal force at C equals the gravitational attraction there.

(*iv*). *Total volume of fluid in the ring*:

The total volume of the ring $= \pi a^2.2\pi c = 2\pi^2a^2c = 2\pi^2.\dfrac{2p_0.c^4}{\varrho_0 GM}$

from (30).

Or

$$V = \frac{4\pi^2 . p_0 . c^4}{\varrho_0 . GM}$$

Or

$$M = \frac{4\pi^2 c^4 . p_0}{G} . \frac{1}{\varrho_0 V}$$

$$= \frac{4\pi^2}{G} . c^4 . p_0 . \frac{1}{m_0} \qquad (32)$$

where m_0 = mass of a fluid of the same volume as the ring but having a density equal to ϱ_0 throughout.

In conclusion, I must thank Prof. N. R. Sen for his encouragement in the preparation of this note.

Ripon College,
Calcutta

Reference

Dive P., (1930), *Rotaitons internes des Astres Fluide.*

ON A TYPE OF NON-STATIONARY TURBULENT WAKE

By

M. Ray

(*Received December 18, 1947*)

The velocity distribution at a considerable distance downstream in the wake behind a symmetrical cylinder has been calculated by Schlichting on the momentum transfer theory and has been extended by Goldstein (1938) for the wake behind a body of revolution on both the momentum transfer and the vorticity transfer theories. The case they considered was when the motion is steady. In the present paper a special type of non-stationary motion has been considered. It is the case of the dying out of a steady wake beginning from the tail upstream. The initial decay factor is given by an exponential term $e^{-\text{const.}\,x}$ and the wake disappears with time.

1. Wake behind a symmetrical cylinder (two dimensional motion)

A cylinder with generators at right angles to the undisturbed flow is placed in a stream, the cylinder having a plane of symmetry parallel to the undisturbed flow. Let x-axis be parallel to the undisturbed flow and in the plane of symmetry, the origin being some unspecified point in the neighbourhood of the cylinder. Let u_0 be the undisturbed stream velocity, and u_0-u, be the mean velocity parallel to the undisturbed flow far downstream. The pressure-gradient may be neglected and the equation for the first approximation to u when (u/u_0) is small, are, on momentum transfer and vorticity transfer theories,

$$\frac{\partial u}{\partial t}+u_0\frac{\partial u}{\partial x}=-\frac{\partial}{\partial y}\left\{l^2\left(\frac{\partial u}{\partial y}\right)^2\right\}, \tag{1P}$$

$$\frac{\partial u}{\partial t}+u_0\frac{\partial u}{\partial x}=-l^2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}, \tag{1T}$$

where l is the mixture length.

We have to solve the above equations subject to the following boundary conditions.

(*a*) Since the wake is symmetrical,

$$\frac{\partial u}{\partial y}=0 \text{ when } y=0, \tag{2}$$

and

(*b*) at the edge of the wake, in order that the velocity may pass smoothly over into that in the main stream,

$$u=0, \text{ and } \frac{\partial u}{\partial y}=0, \text{ when } y=y_0, \tag{3}$$

when $2y_0$ is the breadth of the wake at any section.

In order to solve the above equations, we first make the assumption, due to Prandtl, that l is constant over any one section of the wake, and put

$$u = u(xy).\exp(-\lambda u_0 t), \quad l^2 = l^2(x) \exp(\lambda u_0 t). \tag{4}$$

These substitutions in (1) reduce them to

$$-\lambda u_0 u + u_0 \frac{\partial u}{\partial x} = -2l^2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2}, \tag{5P}$$

and

$$-\lambda u_0 u + u_0 \frac{\partial u}{\partial x} = -l^2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2}. \tag{5T}$$

The two equations are identical except in place of $2l^2$ in (5P) we get l^2 in (5T). Hence both the theories give the same velocity distribution, only the value of l being different. We shall now solve (5P). To do so, we make the following assumptions for $l(x)$ and $u(xy)$, for which we now write simply l, and u respectively.

$$l = \phi(x), \quad \eta = y/\phi(x), \quad \frac{u}{u_0} = \frac{f(\eta)}{\psi(x)}. \tag{6}$$

With these assumptions, (5P) becomes

$$\lambda f(\eta) + \frac{\psi'(x)}{\psi(x)} f(\eta) + \frac{\phi'(x)}{\phi(x)} \eta f'(\eta) = \frac{2}{\phi(x)\psi(x)} f'(\eta) f''(\eta), \tag{7}$$

where dashes denote differentiations with respect to the corresponding arguments.

Let us now put

$$\frac{\psi'(x)}{\psi(x)} = c_1, \quad \frac{\phi'(x)}{\phi(x)} = c_2, \quad \phi(x)\psi(x) = c_3$$

so that

$$\psi(x) = ae^{c_1 x}, \quad \phi(x) = be^{c_2 x} \quad \text{and} \quad abe^{(c_1+c_2)x} = c_3$$

$$\therefore \quad c_1 = -c_2 = -c, \quad \text{say}$$

hence

$$\psi(x) = ae^{-cx}, \quad \phi(x) = be^{cx} \quad \text{and} \quad l = be^{cx}, \quad \eta = \frac{y}{b} e^{-cx}, \quad \frac{u}{u_0} = \frac{f(\eta)}{a} e^{cx}. \tag{8}$$

The equation (7) now becomes

$$(\lambda - c) f(\eta) + c\eta f'(\eta) = \frac{2}{ab} f'(\eta) f''(\eta). \tag{9}$$

We see that $\lambda = 2c$ gives the equation for steady motion. Writing $k^2 = abc$, we obtain, as for steady motion,

$$f(\eta) = \tfrac{1}{9} k^2 \eta_0^3 \{1 - (\eta/\eta_0)^{3/2}\}^2. \tag{10}$$

This gives

$$\frac{u}{u_0} = \frac{c}{9b^2} y_0^3 \{1 - (y/y_0)^{3/2}\}^2 . \exp\{-2c(x + u_0 t)\}, \tag{11}$$

and

$$l^2 = b^2 \exp\{2c(x+u_0t)\} = b^2x.\frac{\exp\{2c(x+u_0t)\}}{x}. \tag{12}$$

This is a solution of the equations of non-steady turbulent wake. It admits probably of one rational interpretation. The expression for u is of the form $u_{\text{stationary}}$ multiplied by the factor $\exp\{-2c(x+u_0t)\}$. This represents the travelling of the stationary condition upstream with velocity u_0. The expression for u shows a possible way of dying out of the turbulent condition which at one stage was steady. If initially at time $t=0$, the extinction factor be represented by a term proportional to $e^{-\text{const.}x}$, formula (11) shows how the turbulence decreases with time and ultimately disappears. The same formula shows that this extinction proceeds upstream with velocity u_0. But this interpretation is defective as y_0 in this scheme initially is proportional to $(y_0)_{\text{stationary}} \times e^{cx}/\sqrt{x}$ and does not depend on time, whereas l^2 increases with time. In any case, it is but a very special type of extinction of the turbulent condition.

There is one test of the correctness of the form of u which has been used by previous authors viz. the calculation of the resistance (drag) from the principle of momentum. The momentum equation can be easily written down in the case of steady motion but in the present state of unsteady motion, this is not easy specially as the pressure distribution in the turbulent wake is not calculable by the usual means.

2. WAKE BEHIND A BODY OF REVOLUTION

When a body of revolution is placed with its axis parallel to the main stream, let x-axis be along the axis of symmetry and r denote the distance from that axis. If u_0 be the undisturbed stream velocity, let u_0-u be the velocity parallel to the axis far downstream in the wake. Then on the momentum transfer theory, the equation for u is

$$\frac{\partial u}{\partial t} + u_0\frac{\partial u}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial r}\left\{l^2 r\left(\frac{\partial u}{\partial r}\right)^2\right\}. \tag{13P}$$

On the vorticity transfer theory, when the turbulence is isotropic, the equation is

$$\frac{\partial u}{\partial t} + u_0\frac{\partial u}{\partial x} = -l^2\frac{\partial u}{\partial r}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\}, \tag{13T1}$$

and when the eddying motion as well as the mean motion, is symmetrical about the axis, the equation is

$$\frac{\partial u}{\partial t} + u_0\frac{\partial u}{\partial x} = -l^2\frac{\partial u}{\partial r}\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r}\right). \tag{13T2}$$

With the assumptions

$$u = u(xr).\exp(-\lambda u_0 t), \quad l^2 = l^2(x).\exp(\lambda u_0 t) \tag{14}$$

and

$$l(x) = \phi(x), \quad \eta = \frac{r}{\phi(x)}, \quad \frac{u(xr)}{u_0} = \frac{f(\eta)}{\psi(x)}, \tag{15}$$

we get for (13P), as in the case of two dimensional motion, the equation in η in the form

$$(\lambda - c)f(\eta) + c\eta f'(\eta) = \frac{1}{ab}\left\{2f'(\eta)f''(\eta) + \frac{f'^2(\eta)}{\eta}\right\}. \tag{16}$$

Here $\lambda = 3c$ gives the equation for steady motion and writing $k^2 = abc$, we get the same equation as (10).

Thus for the momentum transfer theory

$$\frac{u}{u_0} \doteq \frac{1}{9}\,\frac{c}{b^2}\,r_0{}^3.\left\{1 - \left(\frac{r}{r^0}\right)^{3/2}\right\}^2 .\exp\left\{-c(2x + 3u_0t)\right\}. \tag{17}$$

Exactly the same interpretation on the non-stationary character of the wake may be given as in the case of two dimensional motion.

For the equations (13T1) and (13T2) the equations in η are

$$(\lambda - c)f(\eta) + c\eta f'(\eta) = \frac{1}{ab}\left\{f'(\eta)f''(\eta) \pm \frac{f'^2(\eta)}{\eta}\right\}. \tag{18a, b}$$

Here also the problem reduces to that of steady motion with $\lambda = 3c$ and $abc = 1$. The equation (18b) has no real solution while (18a) has a solution in series for small values of η in the form

$$f(\eta) = \alpha - \tfrac{4}{3}\sqrt{(\alpha/3)}.\eta^{3/2} + \tfrac{14}{81}\eta^3 + \cdots. \tag{19}$$

With $\alpha = \frac{1}{9}\eta_0{}^3$, the solution (19) is very near the solution (10). Thus in the inner parts of the wake, the two theories give almost similar solutions, as in the case of steady motion, but in the outer parts Taylor's theory fails to give any correct solution.

Thus the problems of non-stationary wakes in all these cases are reduced to those of stationary wakes, the non-stationary character being exhibited in the velocity distribution by a wave factor.

AGRA COLLEGE,
AGRA

Reference

Goldstein, S , (1938), *Modern Developments in Fluid Dynamics*, Vol. II, 583-592.

FURTHER GENERALIZATION OF THE NOTION OF RELATIVELY PRIME IDEALS

By

Ladislas Fuchs

(Communicated by the Secretary—Received November 11, 1947)

1. Introduction. In a previous paper (Fuchs, 1947) I have introduced the notion of relatively primary ideals as a generalization of that of relatively prime ideals. The definition was as follows: $\mathfrak{g}$ is relatively primary to $\mathfrak{h}$ if a congruence $\mathfrak{m}\mathfrak{g} \equiv 0(\mathfrak{h})$ always involves $\mathfrak{m} \equiv 0(\mathfrak{r})$[$\mathfrak{r}$ denotes the radical of $\mathfrak{h}$].

The present little investigation is devoted to continuing to generalize this notion in such a manner that in the definition we use an arbitrary ideal $\mathfrak{j}$ for the radical and we say: $\mathfrak{g}$ is relatively prime to $\mathfrak{h}$ with respect to $\mathfrak{j}$. After introducing this generalized conception we shall show some of its simple properties. Then it will be inquired under what conditions an ideal $\mathfrak{g}$ is relatively prime to $\mathfrak{h}$ with respect to $\mathfrak{j}$. In finding such conditions one meets some difficulties, since the new notion is fairly general; therefore we shall make certain restrictions which make it possible to get a necessary and sufficient condition. This result yields the main theorem on relatively primary ideals as a special case.

The method dealt with in the sequel is similar to that used in my quoted paper.

2. **Definition**. Let R be a domain of integrity with maximal condition.

The ideal $\mathfrak{g}$ *is called relatively prime to* $\mathfrak{h}$ *with respect to* $\mathfrak{j}$, *if, whenever an ideal* $\mathfrak{m}$ *of* R *is such that* $\mathfrak{m}\mathfrak{g} \equiv 0(\mathfrak{h})$, *we can infer that* $\mathfrak{m} \equiv 0(\mathfrak{j})$, *that is in words: if no product of* $\mathfrak{g}$ *with an ideal* $\mathfrak{m}$ *is contained in* $\mathfrak{h}$ *without* $\mathfrak{m}$ *being contained in* $\mathfrak{j}$.

Throughout the paper we shall symbolically write: $\{\mathfrak{g};\mathfrak{h},\mathfrak{j}\}$ in the place of "$\mathfrak{g}$ *is relatively prime to* $\mathfrak{h}$ *with respect to* $\mathfrak{j}$".

It is worth noticing that $\{\mathfrak{g};\mathfrak{h},\mathfrak{h}\}$ means: $\mathfrak{g}$ is relatively prime to $\mathfrak{h}$ in the common sense, further, if $\mathfrak{r}$ denotes the radical of $\mathfrak{h}$, then $\{\mathfrak{g};\mathfrak{h},\mathfrak{r}\}$ expresses the fact that $\mathfrak{g}$ is relatively primary to $\mathfrak{h}$.

3. **Preliminary Theorems**. Our first purpose is to deduce the simplest properties of this new notion.

Theorem 1. *If there exists an ideal* $\mathfrak{g}$ *at all for which* $\{\mathfrak{g};\mathfrak{h},\mathfrak{j}\}$, *then* $\mathfrak{j}$ *must be a divisor of* $\mathfrak{h}$.

In fact, $\mathfrak{h}\mathfrak{g} \equiv 0(\mathfrak{h})$ implies by hypothesis $\mathfrak{h} \equiv 0(\mathfrak{j})$.

Theorem 2. *If* $\{\mathfrak{g};\mathfrak{h},\mathfrak{j}\}$, *then*

(*a*) *for every divisor* $\mathfrak{g}^*$ *of* $\mathfrak{g}$,
(*b*) *for every multiple* $\mathfrak{g}^*$ *of* $\mathfrak{g}$,
(*c*) *for every divisor* $\mathfrak{j}^*$ *of* $\mathfrak{j}$,

there holds:

$$\{\mathfrak{g}^*;\ \mathfrak{h}^*,\ \mathfrak{j}^*\}$$

For the proof we take into account that

$$\mathfrak{m}\mathfrak{g}^* \equiv 0(\mathfrak{h}^*)$$

involves

$$\mathfrak{m}\mathfrak{g} \equiv 0(\mathfrak{h});$$

thus by hypothesis,

$$\mathfrak{m} \equiv 0(\mathfrak{j})$$

and hence

$$\mathfrak{m} \equiv 0(\mathfrak{j}^*), \qquad \text{Q.E.D.}$$

Let us now pass to the proof of

Theorem 3. *If* $\{\mathfrak{g};\ \mathfrak{h}_1,\ \mathfrak{j}_1\}$ *and* $\{\mathfrak{g};\ \mathfrak{h}_2,\ \mathfrak{j}_2\}$, *then also* $\{\mathfrak{g};\ \mathfrak{h},\ \mathfrak{j}\}$, *where* $\mathfrak{h}$ *is the intersection* [*the least common multiple*] *of* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$, $\mathfrak{h} = \mathfrak{h}_1 \cap \mathfrak{h}_2$, *and* $\mathfrak{j}$ *is that of* $\mathfrak{j}_1$ *and* $\mathfrak{j}_2$, $\mathfrak{j} = \mathfrak{j}_1 \cap \mathfrak{j}_2$.

Since the congruences

$$\mathfrak{m}\mathfrak{g} \equiv 0(\mathfrak{h}) \equiv 0(\mathfrak{h}_i) \quad (i = 1, 2)$$

require

$$\mathfrak{m} \equiv 0(\mathfrak{j}_i) \quad \text{for} \quad i = 1, 2,$$

therefore, indeed, we have

$$\mathfrak{m} \equiv 0(\mathfrak{j})$$

which proves the assertion.

4. **The Main Theorem.** Next we turn to the problem of determining a necessary and sufficient condition in order to decide whether an ideal $\mathfrak{g}$ is or is not relatively prime to $\mathfrak{h}$ with respect to $\mathfrak{j}$. For doing this it is necessary to order an ideal $\mathfrak{j}$ to every ideal $\mathfrak{h}$ of the ring $\mathbf{R}$. By Theorem 1, the ideal $\mathfrak{j}$ ordered to $\mathfrak{h}$ must be a divisor of $\mathfrak{h}$. In addition, for the sake of further investigations, we place a restriction on the system γ of ordered ideals, namely we assume it being subject to the following postulate:

(*A*). *If* $\mathfrak{j}_1$ *and* $\mathfrak{j}_2$ *are ordered to* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *respectively, then let* $\mathfrak{j} = \mathfrak{j}_1 \cap \mathfrak{j}_2$ *be ordered to* $\mathfrak{h} = \mathfrak{h}_1 \cap \mathfrak{h}_2$. That is, the ordered ideal of the intersection of two ideals is the intersection of the two ordered ideals.

Postulate A seems to be no strong restriction, since every interesting special case possesses this property.

Let us notice that some ideals may have the unit ideal as ordered ideal; moreover, the system γ may contain no other ideal than $\mathfrak{o}$!

Now suppose that an arbitrary ideal $\mathfrak{h}$ is represented as the intersection of the ideals $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ in such a manner that

(*I*). *no two of the ideals* $\mathfrak{j}_1, \ldots, \mathfrak{j}_n$ *ordered to the* $\mathfrak{h}_i$ *are equal* [this will be the case as soon as one contracts the components with the same $\mathfrak{j}_i$];

(*II*). *if* $\mathfrak{b}_i = \mathfrak{j}_1 \cap \ldots \cap \mathfrak{j}_{i-1} \cap \mathfrak{j}_{i+1} \cap \ldots \cap \mathfrak{j}_n \not\equiv 0(\mathfrak{j}_i)$, *then* $\mathfrak{a}_i = \mathfrak{h}_1 \cap \ldots \cap \mathfrak{h}_{i-1} \cap \mathfrak{h}_{i+1} \cap \ldots \cap \mathfrak{h}_n$ *is relatively prime to* $\mathfrak{j}_i$, *i.e.*, $\{\mathfrak{a}_i;\ \mathfrak{j}_i,\ \mathfrak{j}_i\}$.

Also this condition may be satisfied after certain contractions.

By (A) it is clear that $\mathfrak{j} = \mathfrak{j}_1 \cap \ldots \cap \mathfrak{j}_n$ is the ideal ordered to $\mathfrak{h}$. Now omit successively all the $\mathfrak{j}_k$ whose omission does not alter their intersection $\mathfrak{j}$. Let the remaining $\mathbf{j}_i$ be denoted by $\mathbf{j}_1^*, \dots, \mathbf{j}_r^*$, (where $r \leq n$) and the components they are ordered to by $\mathbf{h}_1^*, \dots, \mathbf{h}_r^*$. Clearly, $\mathbf{j} = \mathbf{j}_1^* \cap \dots \cap \mathbf{j}_r^*$ and no $\mathbf{j}_s^*$ may be omitted.

Now we are already in a position to formulate and prove our main theorem.

Theorem 4. $\{\mathbf{g}; \mathbf{h}, \mathbf{j}\}$ *if and only if* $\{\mathbf{g}; \mathbf{h}_s^*, \mathbf{j}_s^*\}$ *holds for* $s = 1, \dots, r$.

The proof of the sufficiency is almost evident: Theorem 3 implies by induction that

$$\{\mathbf{g}; \mathbf{h}_1^* \cap \dots \cap \mathbf{h}_r^*, \mathbf{j}_1^* \cap \dots \cap \mathbf{j}_r^*\},$$

i.e., because of $\mathbf{j}_1^* \cap \dots \cap \mathbf{j}_r^* = \mathbf{j}$,

$$\{\mathbf{g}; \mathbf{h}_1^* \cap \dots \cap \mathbf{h}_r^*, \mathbf{j}\};$$

further, Theorem 2a involves owing to $\mathbf{h}_1^* \cap \dots \cap \mathbf{h}_r^* \supseteq \mathbf{h}_1 \cap \dots \cap \mathbf{h}_n = \mathbf{h}$ that

$$\{\mathbf{g}; \mathbf{h}, \mathbf{j}\}.$$

Consequently, the condition is sufficient.

To prove even its necessity, let us assume

$$\mathbf{m}\mathbf{g} \equiv 0(\mathbf{h}_s^*)$$

and take for the sake of convenience $\mathbf{h}_s = \mathbf{h}_s^*$ which can always be attained by a suitable arranging. The evident congruence

$$\mathbf{a}_s\mathbf{g} \equiv 0(\mathbf{a}_s)$$

together with (*) leads us to

$$\mathbf{m}\mathbf{a}_s\mathbf{g} \equiv 0(\mathbf{a}_s \cap \mathbf{h}_s^*) \equiv 0(\mathbf{h}).$$

Hence we conclude, by the hypothesis: $\{\mathbf{g}; \mathbf{h}, \mathbf{j}\}$, that

$$\mathbf{m}\mathbf{a}_s \equiv 0(\mathbf{j}) \equiv 0(\mathbf{j}_s^*).$$

Recall that $\mathbf{j}_s^*$ was assumed not to be omissible in the representation of $\mathbf{j}$, therefore, $\mathbf{b}_i \not\equiv 0(\mathbf{j}_i)$ and so, by requirement II, $\mathbf{a}_s$ is relatively prime to $\mathbf{j}_s^*$. For this reason it follows

$$\mathbf{m} \equiv 0(\mathbf{j}_s^*)$$

which completely establishes the statement.

5. **Applications.** First we shall apply our results to the case when $\mathbf{j}$ is the radical of $\mathbf{h}$, that is, to the theory of relatively primary ideals. In this case $\mathbf{j} = \mathbf{r} \supseteq \mathbf{h}$, and postulate A is clearly satisfied. We only have to examine which representations of $\mathbf{h}$ satisfy I and II. We state that the shortest primary representations do. Indeed, now the $\mathbf{j}_i$ are just the prime radicals of the primary components $\mathbf{g}_i$ and thus I necessarily holds. In addition, if $\mathbf{b}_i \not\equiv 0(\mathbf{j}_i)$ *i.e.*, $\mathbf{j}_i = \mathbf{p}_i$ is a minimal prime ideal of $\mathbf{h}$, then $\mathbf{a}_i$ is relatively prime to $\mathbf{p}_i$, since $\mathbf{a}_i \not\equiv 0(\mathbf{p}_i)$ and $\mathbf{p}_i$ is prime. We see that Theorem 4 says: *$\mathbf{g}$ is relatively primary to $\mathbf{h}$ if and only if it is primary to all the isolated primary components of $\mathbf{h}$.* This is the same result as we have already got in my quoted paper.

Let us now see how our condition looks like in the special case of the relatively prime ideals. Postulate A is now a simple identity; requirement I for the representation of $\mathbf{h}$ is trivially satisfied, while II says that to every component of the representation the intersection of the others is relatively prime. Among the representations of an ideal the relatively-prime-irreducible representation is the most general one which satisfies this condition Thus Theorem 4 expresses the fact that **g** *is relatively prime to* **h** *if and only if* **g** *is relatively prime to each relatively-prime-irreducible component of* **h**. This result is not the best one, since it is not so sharp as that got in the usual way by making use of special properties of relatively prime ideals (*cf.* Waerden, 1940, p. 36-37).

Finally we shall show an example which has not yet been examined. Namely we get a new system of ordered ideals by ordering the unit ideal **o** to every lower* primary ideal and every upper primary ideal to itself As we know, by omitting all the lower primary components of **h** in one of its shortest primary representations, $(\mathbf{h}^{-1})^{-1}$ will be got. Thus in this way the ideal $(\mathbf{h}^{-1})^{-1}$ will be ordered to the ideal **h**. That this system of the so-called **o**-ideals (Krull, 1935, p. 118) satisfies A, is well known. We shall get a representation of **h** satisfying I and II after taking the shortest primary representation of **h** and then contracting all its lower primary components. Applying Theorem 4 for the present case we are led to the following result: **g** *is relatively prime to* **h** *with respect to* $(\mathbf{h}^{-1})^{-1}$ if and only if it is relatively prime to all the upper primary components of **h** with respect to themselves, that is to say, *if and only if* **g** *is relatively prime to* $(\mathbf{h}^{-1})^{-1}$.

Budapest, Hungary.

References

Fuchs, L., (1947), *Kgl. Norske Videnskab. Selskab. Forhand.*, 20 Nr. 7.
Krull, W., (1935), *Idealtheorie, Ergebnisse*, 4, 3, 118.
Waerden, B. L. Van der, (1940), *Moderne Algebra*, 2nd Ed.

---

* B. L. van der Waerden, loc. cit., pp. 93-97 calls them "niederes" resp. "höheres Ideal".

ON AN EXACT SOLUTION OF DIRAC ELECTRON IN THE FIELD OF ELECTROMAGNETIC RADIATION

By

N. D. Sengupta

(Received October 11, 1947)

Introduction

The subject of this paper is to find an exact solution of the Dirac equation of an electron in the field of a plane electromagnetic radiation. A first order approximate solution when the radiation field is monochromatic has already been used by Klein and Nishina to calculate the intensity distribution for Compton effect. The solution here obtained is of the same nature as that was obtained by Volkov (1935) with a special choice of axis. It is interesting to note that four-dimensional generalisation of Volkov's solution follows easily. As in the case of Volkov, here also the solution can be extended to the case where the radiation field is not monochromatic.

Notations

The following notations in four-dimensions will be used throughout the paper.

$\bar{A}$ stands for a 4-vector conjugate to A, whose space-components are opposite to A in sign.

(AB) stands for the scalar product given by

$$(AB) = A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3$$

while $(\bar{A}B)$ and $(A\bar{B})$ are given by

$$(\bar{A}B) = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 = (A\bar{B}).$$

N stands for the four-vector with components

$$(h/c, \quad hl/c, \quad hm/c, \quad hn/c)$$

where l, m, n are direction-cosines of the progressive direction θ which is thus given by

$$\theta = \frac{1}{\hbar}(N\bar{R}) = \frac{2\pi}{c}[ct - lx - my - nz]$$

$$= 2\pi[t - u/c]$$

where $\hbar$ is $1/2\pi$ times Planck's constant h and u stands for $lx + my + nz$.

Further following the convention of the relativity theory the summation symbol will be mostly suppressed. The following abbreviations are also to be noted

$$(A\sigma) \text{ for } A_0+A_1\sigma_1+A_2\sigma_2+A_3\sigma_3\,,$$

$$(\bar{A}\sigma) \text{ or } (A\bar{\sigma}) \text{ for } A_0-A_1\sigma_1-A_2\sigma_2-A_3\sigma_3,$$

where σ's are Pauli matrices.

THE SOLUTION OF THE DIRAC EQUATION

Dirac's Equation of state for an electron in an electromagnetic field can be conveniently be written in the following form

$$\left.\begin{aligned} \frac{\hbar}{i}\left[\frac{1}{c}\frac{\partial}{\partial t}+\sigma_k\frac{\partial}{\partial x_k}\right]\psi-\frac{e}{c}[A_0-A_k\sigma_k]\psi &= -mc\,\varphi \\ \frac{\hbar}{i}\left[\frac{1}{c}\frac{\partial}{\partial t}-\sigma_k\frac{\partial}{\partial x_k}\right]\phi-\frac{e}{c}[A_0+A_k\sigma_k]\phi &= -mc\,\psi \end{aligned}\right\} \tag{1}$$

where σ's are the Pauli matrices, k stands only for the space components. A_0, A_k are the electromagnetic four potential defining the radiation field, which we take to be a plane wave progressing in a fixed direction.

The fact that the field is plane and therefore function of the one-dimentional linear parameter θ suggests the solution to be of the form

$$\left.\begin{aligned} \psi &= P(\theta).\exp\frac{i}{\hbar}S(\theta) \\ \phi &= Q(\theta).\exp\frac{i}{\hbar}S(\theta) \end{aligned}\right\} \tag{2}$$

where S and the matrices P, Q, are function of parameter θ only.

This substitution transforms the original equation (1) to

$$\left.\begin{aligned} \left[\frac{1}{c}\frac{\partial S}{\partial t}+\sigma_k\frac{\partial S}{\partial x_k}\right]P-\frac{e}{c}[A\bar{\sigma}]P+mcQ &= -\frac{\hbar}{i}\left[\frac{1}{c}\frac{\partial P}{\partial t}+\sigma_k\frac{\partial P}{\partial x_k}\right] \\ \left[\frac{1}{c}\frac{\partial S}{\partial t}-\sigma_k\frac{\partial S}{\partial x_k}\right]Q-\frac{e}{c}[A\sigma]Q+mcP &= -\frac{\hbar}{i}\left[\frac{1}{c}\frac{\partial Q}{\partial t}+\sigma_k\frac{\partial Q}{\partial x_k}\right] \end{aligned}\right\} \tag{3}$$

The characteristic equation for S is obtained by neglecting the right hand side of (3) as

$$\left[\frac{1}{c}\frac{\partial S}{\partial t}-\frac{e}{c}A_0\right]^2-\left[\frac{\partial S}{\partial x_k}+\frac{e}{c}A_k\right]^2-m^2c^2 = 0 \tag{4}$$

It is interesting to point out here that (4) is also the Hamiltonian of the classical theory.

Now we can assume

$$S = -(p\bar{R})-\hbar F(\theta) \tag{5}$$

where p is energy momentum four-vector and $F(\theta)$ is a function of θ.

Thus the equation (4) becomes

$$(p\bar{p})+2(p\bar{N}).F'(\theta)+\frac{2e}{c}(p\bar{A})+\frac{2e}{c}(N\bar{A})+(N\bar{N})F'^2(\theta)+\frac{e^2}{c^2}(A\bar{A}) = m^2c^2. \tag{6}$$

But if we remember

$$(N\bar{N}) = 0.$$

Again as the wave is transverse

$$(N\bar{A}) = 0$$

and if

$$(p\bar{p}) = m^2c^2$$

the equation (6) leads to

$$F'(\theta) = -\frac{e}{c}\frac{(p\bar{A})}{(p\bar{N})}-\frac{e^2}{2c^2}\frac{(A\bar{A})}{(p\bar{N})} \tag{7}$$

Now since A is a function of the parameter θ alone, $F(\theta)$ can be obtained by quadrature form (7), as was first shown by Gordon (1927). Coming back to (3) and substituting for S from (5)

$$\left.\begin{aligned}\left\{(p\bar{\sigma})+F'(\theta)(N\bar{\sigma})+\frac{e}{c}(A\bar{\sigma})\right\}P-mcQ &= \frac{1}{i}(N\bar{\sigma})\frac{\partial P}{\partial\theta}\\ \left\{(p\sigma)+F'(\theta)(N\sigma)+\frac{e}{c}(A\sigma)\right\}Q-mcP &= \frac{1}{i}(N\sigma)\frac{\partial Q}{\partial\theta}\end{aligned}\right\} \tag{8}$$

Therefore

$$\left\{(p\sigma)+F'(\theta)(N\sigma)+\frac{e}{c}(A\sigma)\right\}\left\{(p\bar{\sigma})+F'(\theta)(N\bar{\sigma})+\frac{e}{c}(A\bar{\sigma})\right\}P$$

$$-mc\left[mcP+\frac{1}{i}(N\sigma)\frac{\partial Q}{\partial\theta}\right] = \left\{(p\sigma)+F'(\theta)(N\sigma)+\frac{e}{c}(A\sigma)\right\}\frac{(N\bar{\sigma})}{i}\frac{\partial P}{\partial\theta}.$$

We observe the following identities which can be easily verified

$$(p\sigma)(p\bar{\sigma}) = m^2c^2 \tag{9A}$$

$$(p\sigma)(N\bar{\sigma})+(N\sigma)(p\bar{\sigma}) = 2(p\bar{N}) \tag{9B}$$

$$(A\sigma)(N\bar{\sigma})+(N\sigma)(A\bar{\sigma}) = 2(A\bar{N}) = 0 \tag{9C}$$

$$(N\sigma)(N\bar{\sigma}) = (N\bar{N}) = 0 \tag{9D}$$

and

$$2F'(\theta)(p\bar{N})+\frac{e}{c}(p\bar{A})+\frac{e^2}{c^2}(A\bar{A}) = 0 \tag{9E}$$

Thus (8) reduces to

$$2(p\bar{N})\frac{\partial P}{\partial\theta} = (N\sigma)\left\{(p\bar{\sigma})+\frac{e}{c}(A\bar{\sigma})\right\}\frac{\partial P}{\partial\theta} - mc\frac{\partial Q}{\partial\theta}. \tag{10}$$

Again differentiating the first equation of (8)

$$\left\{(p\bar{\sigma})+\frac{e}{c}(A\bar{\sigma})\right\}\frac{\partial P}{\partial\theta}+\frac{e}{c}\left(\frac{\partial A}{\partial\theta}\bar{\sigma}\right)P+F'(\theta)(N\bar{\sigma})\frac{\partial P}{\partial\theta}+F''(\theta)(N\bar{\sigma})P-mc\frac{\partial Q}{\partial\theta} = \frac{1}{i}(N\bar{\sigma})\frac{\partial^2 P}{\partial\theta^2}.$$

from which follows the following relation by making use of the identities (9)

$$2(p\bar{N})\frac{\partial P}{\partial\theta} = -\frac{e}{c}(N\sigma)\left(\frac{\partial A}{\partial\theta}\bar{\sigma}\right)P \tag{11A}$$

Similarly it can be shown that

$$2(p\bar{N})\frac{\partial Q}{\partial\theta} = -\frac{e}{c}(N\bar{\sigma})\left(\frac{\partial A}{\partial\theta}\sigma\right)Q \tag{11B}$$

These variational equations (11A) and (11B) are solvable firstly multiplying (11A) by $(N\bar{\sigma})$ and (11B) by $(N\sigma)$ we get

$$\left.\begin{aligned}(N\bar{\sigma})\frac{\partial P}{\partial\theta} &= 0,\\ (N\sigma)\frac{\partial Q}{\partial\theta} &= 0\end{aligned}\right\} \tag{12}$$

i.e.,

$$(N\bar{\sigma})P = \text{const.} = \mu.$$

Secondly we note

$$(N\sigma)\left(\frac{\partial A}{\partial\theta}\bar{\sigma}\right) = 2\left(\bar{N}\frac{\partial A}{\partial\theta}\right) - \left(\frac{\partial A}{\partial\theta}\sigma\right)(N\bar{\sigma})$$

and

$$\left(\bar{N}\frac{\partial A}{\partial\theta}\right) = 0$$

as a consequence of $(N\bar{A}) = 0$.

Thus

$$2(p\bar{N})\frac{\partial P}{\partial\theta} = \frac{e}{c}\left(\frac{\partial A}{\partial\theta}\sigma\right)\mu$$

or

$$2(p\bar{N})P = \frac{e}{c}(A\sigma)\mu + \mu'$$

i.e.

$$2(p\bar{N})\mu = -\frac{e}{c}(A\bar{\sigma})(N\sigma)\mu + (N\bar{\sigma})\mu'$$

We make use of

$$(N\sigma)\mu = 0$$

to get

$$\mu = \frac{(N\bar{\sigma})}{2(p\bar{N})}\mu'$$

Thus we have

$$P = \left[1 + \frac{e}{2c}\frac{(A\sigma)(N\bar{\sigma})}{(p\bar{N})}\right]P_0 \tag{13A}$$

and similarly

$$Q = \left[1 + \frac{e}{2c}\frac{(A\bar{\sigma})(N\sigma)}{(p\bar{N})}\right]Q_0 \tag{13B}$$

where the constants P_0 and Q_0 are the solutions of the respective equation in the absence of the field, *i.e.*, of the equations

$$\left.\begin{aligned}(p\bar{\sigma})P_0 &= mcQ_0\\ (p\sigma)Q_0 &= mcP_0\end{aligned}\right\} \tag{14}$$

Hence the integral of the Dirac equation (1) is

$$\left.\begin{aligned} \psi &= \exp \frac{i}{\hbar} S. \left[1+ \frac{c}{2c} \frac{(A\sigma)(N\bar{\sigma})}{(p\bar{N})}\right] \psi_0 \\ \phi &= \exp \frac{i}{\hbar} S. \left[1+ \frac{c}{2c} \frac{(A\bar{\sigma})(N\sigma)}{(p\bar{N})}\right] \phi_0 \end{aligned}\right\} \tag{15}$$

The solution of the equations complex-conjugate to (9) are given by

$$\left.\begin{aligned} \psi^{\dagger} &= \psi_0^{\dagger} \left[1+ \frac{c}{2c} \frac{(N\bar{\sigma})(A\sigma)}{(p\bar{N})}\right] \exp\left\{- \frac{i}{\hbar} S\right\} \\ \phi^{\dagger} &= \phi_0^{\dagger} \left[1+ \frac{e}{2c} \frac{(N\sigma)(A\bar{\sigma})}{(p\bar{N})}\right] \exp\left\{- \frac{i}{\hbar} S\right\} \end{aligned}\right\} \tag{16}$$

an interesting feature of the solution.

Let us try to build up the charge and current density corresponding to a fixed value of vector p from the above solutions of the equations, or in other words we find the values of the expressions

$$\psi^{\dagger}(p)\psi(p), \quad \phi^{\dagger}(p)\phi(p)$$

and

$$\psi^{\dagger}(p)\sigma_k\psi(p); \quad \phi^{\dagger}(p)\sigma_k\phi(p)$$

$$\psi^{\dagger}\psi = \psi_0^{\dagger} \left[1+ \frac{e}{2c} \frac{(N\bar{\sigma})(A\sigma)}{(p\bar{N})}\right] \left[1+ \frac{e}{2c} \frac{(A\sigma)(N\bar{\sigma})}{(p\bar{N})}\right] \psi_0$$

$$\psi^{\dagger}\sigma_k\psi = \psi_0^{\dagger} \left[1+ \frac{e}{2c} \frac{(N\bar{\sigma})(A\sigma)}{(p\bar{N})}\right] \sigma_k \left[1+ \frac{c}{2c} \frac{(A\sigma)(N\bar{\sigma})}{(p\bar{N})}\right] \psi_0$$

and similar expressions for ϕ's.

Now if E and H are the electric and the magnetic vectors respectively

$$e_k = \int E_k d\theta = N_k A_0 - A_k N_0$$

and

$$h_k = \int H_k d\theta = A \times N$$

$$(N\sigma)(A\sigma) = -(e_k - ih_k).\sigma_k$$

$$(A\sigma)(N\bar{\sigma}) = -(e_k + ih_k).\sigma_k$$

$$\psi^{\dagger}\psi = \psi_0^{\dagger} \left[1- \frac{c}{c(p\bar{N})} (E_k\sigma_k) + \frac{e^2}{2c^2(N\bar{p})^2} \left(\frac{e_k e_k + h_k h_k}{2} - (P_k\sigma_k)\right)\right] \psi_0$$

$$\psi_0^{\dagger}\sigma_k\psi = \psi_0^{\dagger} \left[\sigma_k - \frac{e}{c} \frac{E_k + (\sigma \times h)_k}{(\bar{p}N)} + \frac{e^2}{2c^2} \frac{P_k}{(N\bar{p})^2} + \frac{e^2}{2c^2} \frac{T_{kl}\sigma_k}{(N\bar{p})^2}\right] \psi_0$$

where P is the Poynting vector and T is Maxwell stress-tensor. These expressions (17) shows that the radiation field may be looked upon as effecting a transformation of the σ and the transformation matrix is given by

$$\left|\begin{array}{cccc} 1+\frac{\lambda^2}{2}T_{11}, & \frac{\lambda^2}{2}T_{12}-\lambda h_3, & \frac{\lambda^2}{2}T_{13}+\lambda h_2, & -\lambda E_1+\frac{\lambda^2}{2}P_1 \\ \frac{\lambda^2}{2}T_{12}+\lambda h_3, & \dots, & \dots, & \dots \\ \dots, & \dots, & \dots, & \dots \\ \lambda E_1-\frac{\lambda^2}{2}P_1, & \dots, & \dots, & 1+\frac{\lambda^2}{2}H \end{array}\right| \tag{18}$$

where $\lambda = \frac{e}{c}\frac{1}{(p\bar{N})}$. The matrix (18) may be simply written as the sum of three matrices

$$I-\lambda F+\lambda^2 T,$$

where F is 4-dimensional antisymmetrical electromagnetic field tensor and T is the 4-dimensional energy momentum tensor.

THE GENERAL SOLUTION

Now as the differential equation (1) is linear, the most general solution is obtained by the principle of superposition.

Thus

$$\left.\begin{aligned} \psi &= \int \exp \frac{i}{\hbar}S.\left[1+\frac{e}{2c}\frac{(A\sigma)(N\bar{\sigma})}{(p\bar{N})}\right]\psi_0(p)dp \\ \phi &= \int \exp \frac{i}{\hbar}S.\left[1+\frac{e}{2c}\frac{(A\bar{\sigma})(N\sigma)}{(p\bar{N})}\right]\phi_0(p)dp \end{aligned}\right\} \tag{19}$$

On account of the relation existing between the energy and momentum namely $(p\bar{p}) = m^2c^2$ the integration is to be extended over the momentum space only. If we restrict ourselves to the positive values of p allowing however p's to assume positive as well as negative values we will have to consider both types of solutions as constituent of the above packet.

CONCLUSION

With the help of the solutions developed here it is possible to study rigorously all the problems relating to the interaction of free electrons and radiation. It is not very much unexpected that a rigorous calculation will lead to some important conclusions as these problems up till now have been studied by approximate methods and also making use of the theory of field quantisation which are not always beyond question.

In conclusion I want to express my thanks to Prof. S. N. Bose for his kind guidance and helpful encouragement throughout the preparation of this paper.

KHAIRA PROFESSOR'S LABORATORY
DEPARTMENT OF PURE PHYSICS
CALCUTTA UNIVERSITY

References

Gordon, (1927), *Zeit f Physik*, **40**, 117.
Volkov, (1935), *Zeit. f Physik*, **94**, 250.
———, (1936), *C R U. S S R.*, 1, 605.

ON THE INADEQUACY OF MEASURING THE PEAKEDNESS OF A DISTRIBUTION CURVE BY THE STANDARDISED FOURTH MOMENT

BY

M. C. CHAKRABARTI

(Communicated by N. M. Basu—Received December 20, 1947)

1. The common practice of measuring the peakedness of a distribution curve in relation to the corresponding normal curve by the standardised fourth moment is known to be defective. Recently Kaplansky (1945) has given illustrations demonstrating this fact but owing to long usage, the practice has become so deep rooted that the examples cited may be looked upon by practical statisticians as far-fetched and pathological in character. The object of this note is to demonstrate by distribution curves of statistics coming from well-known statistical populations that the peakedness of a distribution curve in relation to the corresponding normal curve is independent of the sign of $(\beta_2 - 3)$ where β_2 stands for the standardised fourth moment. It is believed that the examples given below have not been discussed in the present context anywhere in print.

2. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a rectangular population: $p(x) = R^{-1}$, $\theta - R/2 \leq x \leq \theta + R/2$. Let the minimum observation be denoted by u and the maximum observation by v; then the distribution law of the statistic $t = (u+v)/2$ is given by

$$p(t) = (n/R^n)[R - 2\,|\,t-\theta\,|]^{n-1}, \quad \theta - R/2 \leq t \leq \theta + R/2 \qquad (1)$$

This is a symmetrical distribution about the mean θ and all its odd moments about the mean are zero. The $(2k)$th moment about the mean

$$\mu_{2k} = R^{2k} \Big/ \binom{2k+n}{2k}, \qquad k = 0, 1, 2, \ldots$$

whence

$$\beta_2 = \frac{6(n+1)(n+2)}{(n+4)(n+3)} = 6\left[1 - \frac{4n+10}{(n+4)\,(n+3)}\right]$$

Now

$$\frac{4n+10}{(n+4)(n+3)} \geq \tfrac{1}{2} \qquad \text{if } 8 + n - n^2 \geq 0$$

i.e., if $n \leq (\sqrt{33}+1)/2$.

The modal ordinate of a normal curve having $\sigma^2 = \dfrac{R^2}{2(n+1)(n+2)}$ is

$$R^{-1}\{(n+1)(n+2)/\pi\}^{\frac{1}{2}} = nR^{-1}\{[1 + n^{-2}(3n+2)]/\pi\}^{\frac{1}{2}} < nR^{-1} \quad \text{for } n \geq 2.$$

The cases $n=2$ and $n=3$ provide, therefore, examples of distribution curves more peaked than the corresponding normal curves although having negative kurtosis ($\beta_2-3<0$).

When n is given non-integral values, the statistic t loses its old physical meaning but (1) may still be regarded as the equation of a distribution curve. The following two cases are of special interest:

(*i*) $n=\dfrac{\sqrt{33}+1}{2}$: In this case $\beta_2=3$, but the distribution curve is more peaked than the corresponding normal curve.

(*ii*) $n=\dfrac{3+\sqrt{(1+8\pi)}}{2(\pi-1)}$: In this case $\beta_2-3<0$, but the distribution curve and the corresponding normal curve are equally peaked at the mean (mode).

3. Rietz (1936) has considered a set of points (x, y) uniformly distributed over the parallelogram whose corner points are (a_1, b_1), (a_2, b_3), $(b_2, b_2+b_3-b_1)$ and (a_1, b_2) where $a_2>a_1>0$ and $b_3>b_1$. The regression of y on x is linear and the line of regression is given by

$$y=\frac{b_3-b_1}{a_2-a_1}(x-a_1)+\frac{b_1+b_2}{2}$$

Let

$$t=\left\{y-\frac{b_3-b_1}{a_2-a_1}(x-a_1)-\frac{b_1+b_2}{2}\right\}\Big/x$$

Then

$$p(t)=\begin{cases}\dfrac{a_2+a_1}{2(b_2-b_1)}, \quad \text{when } -\dfrac{b_2-b_1}{2a_2}\leq t\leq\dfrac{b_2-b_1}{2a_2},\\[2ex] \dfrac{1}{2(a_2-a_1)(b_2-b_1)}[(b_2-b_1)^2/(4t^2)-a_1^2],\\[2ex] \qquad\qquad \text{when either } \dfrac{b_3-b_1}{2a_1}\leq t\leq\dfrac{b_2-b_1}{2a_1} \text{ or } -\dfrac{b_3-b_1}{2a_1}\leq t\leq\dfrac{b_3-b_1}{2a_2}.\end{cases}$$

This is a symmetrical distribution about the mean zero and its odd moments about the mean are all zero. Denoting by μ_2 and μ_4, the second and fourth moments about the mean, we get easily

$$\mu_2=(b_2-b_1)^2/(12a_1a_2)$$

$$\mu_4=\{(b_2-b_1)^4/(24a_1^3a_2^3)\}(a_2^2+a_1a_2+a_1^2)$$

whence

$$\beta_2=\frac{3}{5}\cdot\frac{a_2^2+a_1a_2+a_1^2}{a_1a_2}.$$

The following three cases are of special interest:

(*i*) $a_2=(2+\sqrt{3})a_1$: In this case $\beta_2=3$. The ratio of the modal ordinate to the ordinate of the corresponding normal curve is $2/\sqrt{\pi}>1$ We have, therefore, an interesting distribution curve for which $\beta_2=3$ but which is less peaked than the corresponding normal curve.

(ii) $a_2 = a_1[(12/\pi-1)+\sqrt{\{(12/\pi-1)^2-1\}}]$: In this case $\beta_2-3>0$, but the distribution curve has a peak equal to that of the corresponding normal curve.

(iii) $a_2 = ca_1$, $2+\sqrt{3} < c < (12/\pi-1)+\sqrt{\{(12/\pi-1)^2-1\}}$: In this case $\beta_2-3>0$, but the distribution curves are less peaked than the corresponding normal curves.

4. Let us denote by y_M the modal ordinate of a distribution curve and by y_N the ordinate of the corresponding normal curve. We have seen that there are distribution curves for which

$$\begin{array}{ll} \beta_2 < 3, & y_N = y_M; \\ \beta_2 < 3, & y_N < y_M; \\ \beta_2 = 3, & y_N > y_M; \\ \beta_2 = 3, & y_N < y_M; \\ \beta_2 > 3, & y_N > y_M; \\ \beta_2 > 3, & y_N = y_M; \end{array}$$

As cases where (i) $\beta_2 < 3$ and $y_N > y_M$ (ii) $\beta_2 > 3$ and $y_N < y_M$, abound in statistical literature, they are not specially cited here. We thus see that from a knowledge of β_2 we cannot determine the relationship between y_M and y_N.

DEPARTMENT OF MATHEMATICS
AND STATISTICS,
DACCA UNIVERSITY.

References.

Kaplansky, I.. (1945), *Jour. Am. Stat. Assoc*, **40**, 259.
Rietz, H. L., (1936), *Ann. Math. Stat.*, **7**, 150-52,

ON ZEROS OF QUASI-ANALYTIC (B) FUNCTIONS

BY

Z. ZAHORSKI

(Communicated by the Secretary—Received November 20, 1947)

We call a function $f(z)$, defined for $z \in Z$, where Z is a set of complex numbers, *quasi-analytic* $(B\varepsilon_n)$ on Z, (where $\{\varepsilon_n\}$ is a sequence of positive numbers, $\lim\limits_{n \to \infty} \varepsilon_n = 0$) when a sequence of natural numbers $\{n_k\}$, and a sequence of polynomials $\{P_k(z)\}$ can be chosen so that $P_k(z)$ is of degree n_k, $n_k \to +\infty$ and

$$|f(z) - P_k(z)| \leq \varepsilon_{n_k} \text{ for every } z \in Z \text{ and every } k.$$

We define the derivative of $f(z)$ on Z as the limit of the differential ratios between a constant and a variable point, both belonging to Z, when their distance tends to 0. The derivatives of higher degrees are defined in an analogous manner. When Z is the interval $[0, 1]$ or the circle $|z| \leq 1$, then the following theorem is valid: the convergence of a sequence of functions and the uniform convergence of the sequence of derivatives implies that the limit of the sequence of derivatives is equal to the derivative of the limit of the sequence of functions.

In particular, a quasi-analytic function $(B\delta^n)$, $0 < \delta < 1$ is called quasi-analytic (B) (in the sense of Bernstein).

M. St Mazurkiewicz and Miss H. Szmuszkowiczowna (1935, 1937) had proved, that a quasi-analytic (B) function in $[0, 1]$ vanishes identically, when it vanishes on a set of transfinite diameter > 0, the more when it vanishes on a set of positive measure. It does not necessarily vanish, when it has derivatives of all degrees $= 0$, even when it vanishes with them on a set of points of continuum-power. The purpose of this work is to construct an analogous example in the circle $|z| \leq 1$, with some generalizations. We shall namely prove the theorem:

For every sequence of positive numbers $\{\varepsilon_n\}$ *and for every perfect set* E *of points* z, *fulfilling* $|z| = 1$, *there exists a function* $f(z)$ *quasi-analytic* $(B\varepsilon_n)$ *in* $|z| \leq 1$, *possessing all derivatives quasi-analytic* $(B\varepsilon_n)$, *that does not vanish identically and such that*

$$f(z) = f^{(n)}(z) = 0$$

for every n *and every* $z \in E_1 \subset E$, *where* E_1 *is of continuum-power.*

The present work follows the lines of M. Mazurkiewicz and M. Szmuszkowiczowna with the exception of Lemma 1, the proof of which in the work of M. M. Mazurkiewicz-Szmuszkowiczowna is based on the theorem of Weierstrass about the uniform approximation of a continuous function by polynomials in $[0, 1]$. The analogous proof in the complex domain, where this theorem cannot be applied, presents serious difficulties. The lemma itself differs somewhat from the corresponding one of M. M. Mazurkiewicz-Szmuszkowiczowna and its proof is based on the properties of the function $\exp(-\lg^2 az)$.

Lemma 1. *For every pair of natural numbers k, n, a complex number z_0, $|z_0| = 1$, and every positive number ε there exist a polynome $P(z)$ of degree $> n$ and a positive number $\varepsilon' < \varepsilon$, such that for $|z| \leq 1$,*

$$P(z_0) = 1$$

$$|P^{(m)}(z)| < \varepsilon' \text{ for } |z - z_0| \geq \varepsilon', \quad m = 0, 1, 2, \ldots, k, \quad (P^{(0)}(z) = P(z)) \tag{1}$$

$$|P^{(m)}(z)| < M(k)(1/\varepsilon')^{m+1} \text{ for } |z - z_0| < \varepsilon', \quad m = 0, 1, 2, \ldots, k, \quad M(k) = 2^{2k+2}.k!\exp(\pi^2/4). \tag{2}$$

Proof. For the function $y(z) = \exp(-\lg^2 az)$, $a > 0$, regular for $\mathcal{R}(z) > 0$, $\lg az = \lg|az| + i\theta$, $-\pi/2 < \theta < \pi/2$ it can be proved by induction:

$$y^{(m)}(z) = \frac{\exp(-\lg^2 az)}{z^m}(\alpha_{m,m}\lg^m az + \alpha_{m,m-1}\lg^{m-1}az + \ldots + \alpha_{m,0}) \tag{3}$$

where $\alpha_{m,p}$ are whole-numbers, defined completely by m, p, $\alpha_{0,0} = 1$.

We denote

$$M_k = \exp(\pi^2/4).2^k \max_{0 \leq m \leq k} \sum_{p=0}^{m} |\alpha_{m,p}| \tag{4}$$

and choose

$$\varepsilon_1 < \min\left(\tfrac{1}{2}\varepsilon, 1, \{M_k^2 \exp(-\pi^2/2)\}^{1/(3+2k)}\right),$$

so small that

$$-\frac{1}{3k+2k^2}\lg M_k - \frac{1}{3k+2k^2}\lg\left(\sqrt[2]{\varepsilon_1^{3+2k}}/M_k\right) - \sqrt[2]{\{-\lg(\sqrt[2]{\varepsilon_1^{3+2k}}/M_k)\}} > 0, \quad \varepsilon_1 > 0 \tag{5}$$

what is possible, for in

$$\frac{1}{3k+2k^2}\chi^2 - \chi - \frac{\lg M_k}{3k+2k^2},$$

the co efficient of χ^2 is positive, so that the inequality (5) is fulfilled by every sufficiently great $\chi = \sqrt{\{-\lg(\sqrt{\varepsilon_1^{3+2k}}/M_k)\}}$, *i.e.*, by every sufficiently small ε_1. We lay down

$$a = \varepsilon_1^{-1-1/2k} \tag{6}$$

then $a > 1$ and denoting

$$\varrho = \exp\left[\{1 + (1/2k)\}\lg \varepsilon_0 + \sqrt{\{-\lg(\sqrt{\varepsilon_1^{3+2k}}/M_k)\}}\right]$$

we have

$$\varrho = (1/a)\exp\{\sqrt{(-\lg(\varepsilon_1/M_k a^k))}\} > (1/a)\exp(\pi/2) \tag{7}$$

for

$$\varepsilon_1 : a^k = \varepsilon_1 : \varepsilon_1^{-k-\frac{1}{2}} = \varepsilon_1^{1\frac{1}{2}+k} = \sqrt[2]{\varepsilon_1^{3+2k}} < \sqrt[2]{\{M_k^2 \exp(-\pi^2/2)\}} = M_k \exp(-\pi^2/4),$$

$$\varepsilon_1/(M_k a^k) < \exp(-\pi^2/4), \quad -\lg\{\varepsilon_1/(M_k a^k)\} > \pi^2/4.$$

From (5), (6) it follows

$$-\frac{1}{3k+2k^2}\lg\left(\sqrt[2]{\varepsilon_1^{3+2k}}\right) > \sqrt[2]{\{-\lg(\sqrt[2]{\varepsilon_1^{3+2k}}/M_k)\}} = \sqrt{(-\lg(\varepsilon_1/a^k M_k))}$$

$$\varepsilon_1^{-1/2k} > \exp\sqrt{\{-\lg(\varepsilon_1/M_k a^k)\}}, \quad \varepsilon_1 > \varrho \tag{8}$$

From (3) it follows for $y^{(m)}(z)$ by so chosen a and $|z| \geq \varepsilon_1$:

$$|y^{(m)}(z)| < \frac{|\exp(-\lg^2 az)|}{|z^m|} \sum_{p=0}^{m} |\alpha_{m,p}| \cdot |\lg az|^p$$

$$= \frac{\exp(-\lg^2|az|+\theta^2)}{|z|^m} \sum_{p=0}^{m} |\alpha_{m,p}| \cdot |\lg|az|+i\theta|^p$$

$$< \frac{\exp(\pi^3/4-\lg^2|az|)}{|z|^m} \sum_{p=0}^{m} |\alpha_{m,p}| \cdot (\log|az|+\pi/2)^m$$

and denoting $|z| = \chi \geq \varepsilon_1 > \varrho > (1/a)\exp(\pi^3/2)$ according to (7) and (8), we have

$$\lg a\chi > \pi/2 \quad \lg a\chi + \pi/2 < 2\lg a\chi,$$

and hence

$$|y^{(m)}(z)| < \{\exp(\pi^2/4).\sum_{p=0}^{m} |\alpha_{m,p}|\}.\frac{\exp(-\lg^2 a\chi)}{(a\chi)^m} a^m 2^m \lg^m a\chi \tag{9}$$

$$\leq a^k \exp(-\lg^2 a\chi).\left(\frac{\lg a\chi}{a\chi}\right)^m.2^k.\exp(\pi^2/4) \sum_{p=0}^{m} |\alpha_{m,p}|$$

for $m = 0, 1, 2, \ldots, k$. $(\lg t)/t$ having a maximum for $t = e$, equal to $e^{-1} < 1$, we can put in (9) 1 instead of $(\lg a\chi)/(a\chi)$ as $a\chi > \exp(\pi/2) > 1$ and it follows from (4):

$$|y^{(m)}(z)| < M_k a^k.\exp(-\lg^2 a\chi) \quad \text{for} \quad |z| \geq \varepsilon_1, \; m = 0, 1, 2, \ldots, k. \tag{10}$$

$\lg^2 a\chi$ increasing for $a\chi \geq 1$, we have, as $a\chi > a\varrho > \exp(\pi/2) > 1$,

$$|y^{(m)}(z)| < M_k a^k.\exp(-\lg^2 a\varrho) = M_k a^k.\exp\{\lg(\varepsilon_1/M_k a^k)\} = \varepsilon_1 \tag{11}$$

for $|z| \geq \varepsilon_1 > \varrho$, according to (7), (10), for $m = 0, 1, 2, \ldots, k$.

We evaluate $y^{(m)}(z)$ in the circle $|z-1-1/a| \leq 1$, according to the formula of Cauchy, $y^{(m)}(z) \leq m!.\mathrm{Max.}|y(z)|/r^m$ where r denotes the radius of a circle with the centre z, lying in the domain of regularity and Max. is taken on the perimeter of the circle. For $\mathcal{R}z > 0$ we have $|y(z)| < \exp(\pi^2/4)$ and in the circle considered we can take $r = 1/2a$.

$$|y^{(m)}(z)| < \frac{k!\exp(\pi^2/4)}{(1/2a)^m} \leq k!\,2^k.a^m.\exp(\pi^2/4), \quad \text{for} \quad m = 0, 1, 2, \ldots, k.$$

$$|y^{(m)}(z)| < k!\,2^k.\exp(\pi^2/4).\varepsilon_1^{-m-m/2k} < k!\,2^k.\exp(\pi^2/4).\varepsilon_1^{-m-1}, \quad \text{for} \quad m = 0, 1, 2, \ldots, k, \tag{12}$$

The points of the circle $|z-1-1/a| \leq 1$, for which $|z-1/a| \geq \varepsilon_1$, are lying outside the circle $|z| \leq \varepsilon_1$, the more the points $|z-1/a| \geq 2\varepsilon_1$. Denoting $2\varepsilon_1 = \varepsilon'$, we obtain from (11) and (12):

$$|y^{(m)}(z)| < \varepsilon_1. \text{ for } |z-1/a| \geq 2\varepsilon_1, \; |z-1-1/a| \leq 1, \; m = 0, 1, 2, \ldots, k$$

$$|y^{(m)}(z)| < (1/\varepsilon_1^{m+1})k!\,2^k \exp(\pi^2/4), \text{ for } |z-1/a| < 2\varepsilon_1, \; |z-1/a-1| \leq 1, \; m = 0, 1, 2, \ldots, k.$$

The function $y(z)$ can be developed in a power series from the point $1+1/a$, the radius of convergence being $1+1/a$, the series converging uniformly in the circle

$|z-1-1/a|\leq 1$ with all the derivatives. $y(z)$ is not a polynomial and therefore a polynomial $W_1(z)$ of degree $> n$, (a segment of the power series) exists, such that

$$|y(z)-W_1(z)|<\varepsilon_1/2, \quad \text{for} \quad |z-1-1/a|\leq 1$$

$$|y^{(m)}(z)-W_1^{(m)}(z)|<\varepsilon_1 \quad \text{for} \quad |z-1-1/a|\leq 1, \quad m=1,2,\ldots,k.$$

The polynomial $W(z)=1-W_1(1/a)+W_1(z)$, fulfills, in value $y(1/a)=1$, the conditions:

$$|y^{(m)}(z)-W^{(m)}(z)|<\varepsilon_1, \quad m=0,1,2,\ldots,k.$$

$$|W^{(m)}(z)|<2\varepsilon_1, \quad \text{for} \quad |z-1/a|\geq 2\varepsilon_1, \quad |z-1-1/a|\leq 1, \quad m=0,1,2,\ldots,k.$$

$$|W^{(m)}(z)|<\varepsilon_1+(1/\varepsilon_1^{m+1})k!\,2^k\exp(\pi^2/4)<2(1/2\varepsilon_1)^{m+1}.k!\,2^k\exp(\pi^2/4).2^{m+1}$$

$$\leq (1/2\varepsilon_1)^{m+1}.2^{2k+2}.\exp(\pi^2/4).k! = M(k)/(\varepsilon')^{m+1}$$

for $|z-1/a|<2\varepsilon_1$, $|z-1-1/a|\leq 1$, $m=0,1,2,\ldots,k$ and $W(1/a)=1$. Therefore the polynomial $P(z)=W(-z/z_0+1+1/a)$ fulfills all the conditions of the Lemma 1 $(2\varepsilon_1<\varepsilon)$.

Lemma 2. *For every polynomial $W(z)$, possessing in $z_0, z_1, \ldots, z_n$ on the perimeter of the circle $|z|\leq 1$ zeros of degrees $k_0\geq 2, k_1, k_2, \ldots, k_n$ and for every pair of numbers $\varepsilon(>0)$ and N (natural), there exists a polynome $W_1(z)$ of degree $>N$, possessing zeroes of degrees $\geq k_0+1, k_1, k_2, \ldots, k_n$ in $z_0, z_1, z_2, \ldots, z_n$, such that*

$$|W^{(m)}(z)-W_1^{(m)}(z)|<\varepsilon, \quad \text{for} \quad |z|\leq 1, \quad m=0,1,2,\ldots,k_0-2.$$

Proof. If $W(z_0)=W'(z_0)=\ldots=W^{(k_0-1)}(z_0)=W^{(k_0)}(z_0)=0$ and $W(z)$ is of degree $>N$, it suffices to assume $W_1(z)=W(z)$. If $W^{(k_0)}(z_0)=0$ and $W(z)$ is of degree $\leq N$, then we lay down

$$W_1(z)=W(z)+\alpha(z-z_0)^{N+1}(z-z_1)^{k_1}\ldots(z-z_n)^{k_n}$$

where

$$0<\alpha<\varepsilon.[\max_{|z|\leq 1}\{|W_2(z)|, |W_2'(z)|, \ldots, |W_2^{(k_0-2)}(z)|\}]^{-1},$$

and

$$W_2(z)=(z-z_0)^{N+1}(z-z_1)^{k_1}\ldots(z-z_n)^{k_n}.$$

It is easy to verify that $W_1(z)$ fulfills the conditions of Lemma 2. If $W^{(k_0)}(z)\neq 0$, then it is necessary and sufficient to construct a polynomial $W_3(z)$ of degree greater than max (degree of $W(z)$, N), which would have in z_0 zero of degree k_0 and a given value of k_0-th derivative $=-W^{(k_0)}(z_0)=D$ and in $z_1, z_2, \ldots, z_n$ zeros of degrees $k_1, k_2, \ldots, k_n$ such that $|W_3^{(m)}(z)|<\varepsilon$, for $|z|\leq 1$, $m=0,1,2,k_0-2$. I denote

$$W_3(z)=(z-z_0)^{k_0}(z-z_1)^{k_1}\ldots(z-z_n)^{k_n}P(z), \quad P(z_0)\neq 0,$$

then

$$W_3^{(k_0)}(z_0)=(z_0-z_1)^{k_1}(z_0-z_2)^{k_2}\ldots(z_0-z_n)^{k_n}P(z_0).k!$$

and it suffices that

$$P(z_0)=\frac{D}{k_0!(z_0-z_1)^{k_1}\ldots(z_0-z_n)^{k_n}}=\frac{d}{(k_0-1)!} \tag{31}$$

Denoting $(z-z_0)^{k_0}(z-z_1)^{k_1}\ldots(z-z_n)^{k_n} = W_4(z)$, we lay down

$$W_4^{(m)}(z) = (z-z_0)^{k_0-m}\,\overline{W}_m(z), \quad m = 0, 1, 2, \ldots, k_0-1, \tag{14}$$

$$M = \max|\overline{W}_m(z)|, \quad \text{for} \quad z \leq 1, \quad m = 0, 1, 2, \ldots, k_0-1 \tag{15}$$

$$\varepsilon_0 < \min\{1, \varepsilon/(|d|M.2^{3k_0+2})\}, \quad \varepsilon_0 > 0. \tag{16}$$

We choose a polynomial $p(z)$, fulfilling the conditions of Lemma 1, for $k = k_0-1$, $\varepsilon = \varepsilon_0$, of degree higher than max $(N$, degree of $W(z)) = N_1$ and we lay down

$$P(z) = p(z).d/(k_0-1)!,$$

then the condition (13) is fulfilled and

$$W_3^{(m)}(z) = \frac{d}{(k_0-1)!}\sum_{l=0}^{m} W_4^{(m-l)}(z)p^{(l)}(z).\frac{m!}{l!(m-l)!} \tag{17}$$

For $|z-z_0| \geq \varepsilon'$, $z \leq 1$, we obtain from (17), (1) and from (14), (15), in value of $|z-z_0| \leq 2$,

$$|W_3^{(m)}(z)| < \frac{|d|M}{(k_0-1)!}\sum_{l=0}^{m} 2^{k_0-m+l}.\varepsilon'.\frac{m!}{l!(m-l)!}$$

$$\leq \frac{2^{k_0}\varepsilon'|d|M}{(k_0-1)!}\sum_{l=0}^{m}\frac{m!}{l!(m-l)!} = \frac{2^{k_0+m}\varepsilon'|d|M}{(k_0-1)!}$$

$$|W_3^{(m)}(z)| < \frac{2^{2k_0-1}\varepsilon_0|d|M}{(k_0-1)!} < \varepsilon.\frac{1}{(k_0-1)!\,2^{k_0+3}} < \varepsilon \quad \text{for} \quad m = 0; 1, 2, \ldots, k-1 \tag{18}$$

according to (16). For $|z-z_0| < \varepsilon'$, $|z| \leq 1$ it follows from (17), (2), (14), (15); for $m = 0, 1, 2, \ldots, k_0-1$

$$|W_3^{(m)}(z)| < \frac{|d|M}{(k_0-1)!}\sum_{l=0}^{m}(\varepsilon')^{k_0-m+l}(1/\varepsilon')^{l+1}M(k_0-1).\frac{m!}{l!(m-l)!}$$

$$= (\varepsilon')^{k_0-m-1}.2^{2k_0}(k_0-1)!.\exp(\pi^2/4).\frac{|d|M}{(k_0-1)!}2^m,$$

$$|W_3^{(m)}(z)| < \varepsilon' 2^{3k_0-2}.\exp(\pi^2/4).|d|M$$

$$< \varepsilon_0.2^{3k_0+2}|d|M \quad \text{for} \quad m = 0, 1, 2, \ldots, k_0-2, \tag{19}$$

for $\varepsilon' < \varepsilon_0 < 1$, $(\varepsilon')^{k_0-m-1} \leq \varepsilon'$,

$$\exp(\pi^2/4) < \exp(1.58)^2 < \exp(2.5) < 12.5 < 2^4.$$

From (16) we obtain

$$|W_3^{(m)}(z)| < \varepsilon, \quad \text{for} \quad m = 0, 1, 2, \ldots, k_0-2, \quad |z-z_0| < \varepsilon', \quad |z| \leq 1$$

$$|W_3^{(m)}(z)| < \varepsilon, \quad \text{for} \quad m = 0, 1, 2, \ldots, k_0-2, \quad |z| \leq 1$$

according to (18) and (19). Hence the polynomial $W_1(z) = W(z) + W_3(z)$ fulfills all the conditions of Lemma 2.

Lemma 3. *For every polynomial $W(z)$ possessing in $z_0, z_1, \dots, z_n$ zeros of degree $\geq k \geq 2$, where $z_i \in E$, $i = 0, 1, 2, \dots, n$, where E is a perfect set of points of the perimeter of the circle $|z| \leq 1$ and for every natural N and $\varepsilon > 0$, there exists a polynomial $W_1(z)$ of degree $> N$, possessing in $z_0, z_0', z_1, z_1', \dots, z_n, z_n'$ zeros of orders $\geq k+1$, and besides* $z_i' \in E$ for $i = 0, 1, 2, \dots, n$, $z_i' \neq z_k'$ for $i \neq k$, $z_i' \neq z_k$,

$$|z_i' - z_i| < \varepsilon$$

$$|W_1^{(m)}(z) - W^{(m)}(z)| < \varepsilon \quad \textit{for} \quad |z| \leq 1, \quad m = 0, 1, 2, \dots, k-2. \tag{20}$$

Proof. We apply the Lemma **2** to $z_0, z_1, \dots, z_n$ $[\varepsilon/\{2(n+1)(k+2)\}$ instead of $\varepsilon]$ and we obtain a polynomial $W_*(z)$, which has in $z_0, z_1, \dots, z_n$ zeros of degree $\geq k+1$, and such that:

$$|W_*^{(m)}(z) - W^{(m)}(z)| < \frac{\varepsilon}{2(k+2)} \quad \text{for} \quad m = 0, 1, 2, \dots, k-2, \quad |z| \leq 1,$$

We apply the same procedure to $W_*(z)$, etc., $k+2$ times and finally we obtain a polynomial $W_{**}(z)$, which has in $z_0, z_1, \dots, z_n$ zeros of degree $\geq 2k+2$, and such that

$$|W_{**}^{(m)}(z) - W^{(m)}(z)| < \varepsilon/2 \quad \text{for} \quad m = 0, 1, 2, \dots, k-2, \quad |z| \leq 1. \tag{21}$$

We lay down

$$W_{**}(z) = [(z-z_0)(z-z_1)(z-z_2)\dots(z-z_n)]^{2k+2}\overline{W}(z),$$

$$W_1(z, z_0', z_1', \dots, z_n') = [(z-z_0)(z-z_0')(z-z_1)(z-z_1')\dots(z-z_n)(z-z_n')]^{k+1}\overline{W}(z)$$

then

$$\lim_{\substack{z_0' \to z_0 \\ z_1' \to z_1 \\ \dots\dots\dots \\ z_n' \to z_n}} W_1^{(m)}(z, z_0', z_1', z_2', \dots, z_n') = W_{**}^{(m)}(z) \quad \text{for} \quad m = 0, 1, 2, \dots, k-2, \quad |z| \leq 1$$

uniformly with respect to z. We surround the points $z_0, z_1, \dots, z_n$ with circles with radii $< \varepsilon$ so small, that they are mutually disjoint and choosing in these circles $z_i' \neq z_i$, $i = 0, 1, 2, \dots, n$, we have:

$$|W_1^{(m)}(z, z_0', z_1', \dots, z_n') - W_{**}^{(m)}(z)| < \varepsilon/2 \quad \text{for } m = 0, 1, 2, \dots, k-2 \quad |z| \leq 1, \tag{22}$$

(we choose z_i' in the circle of centre z_i). The set E being dense in itself we can choose $z_i' \in E$. By z_i' so chosen we denote $W_1(z) = W_1(z, z_0', z_1', \dots, z_n')$, from (22) and (21) follows (20), and $W_1(z)$ fulfills all the conditions of Lemma 3, for, according to Lemma 2, the degree of $W_1(z) =$ degree of $W_{**}(z)$ can be $> N$.

Lemma 4. *For every sequence of positive numbers $\{\varepsilon_n\}$ there exists a sequence of positive numbers $\{\varepsilon_n'\}$ such that for every n, m*

$$\varepsilon_n \geq \varepsilon_n', \quad \varepsilon_m' \geq \sum_{i=m+1}^{\infty} \varepsilon_i'.$$

Proof. We define the sequence $\varepsilon_n^{(1)} = \min(\varepsilon_n, \varepsilon_1/2^{n-1})$ and, by means of induction, sequences $\varepsilon_n^{(k)} = \min(\varepsilon_n^{(k-1)}, \varepsilon_k^{(k-1)}/2^{n-k})$, involving only positive numbers. These sequences possess the property $\varepsilon_n^{(k)} \leq \varepsilon_n^{(k-1)} \leq \dots \leq \varepsilon_n^{(r)} \leq \varepsilon_n$ $(1 \leq p < k)$, $\varepsilon_m^{(m-1)} = \varepsilon_m^{(m)}$. We put $\varepsilon_n' = \varepsilon_n^{(n)}$; this sequence fulfills the conditions of the Lemma 4, for

$$\varepsilon_n^{(n)} \leq \varepsilon_n, \quad \sum_{n=m+1}^{\infty} \varepsilon_n^{(n)} \leq \sum_{n=m+1}^{\infty} \varepsilon_n^{(m)} \leq \sum_{n=m+1}^{\infty} \frac{\varepsilon_m^{(m-1)}}{2^{n-m}} = \varepsilon_m^{(m-1)} = \varepsilon_m^{(m)}.$$

Proof of the Theorem (construction): Let E denote a perfect set of points of the perimeter of the circle $|z| \leq 1$, and let $\{\varepsilon_n\}$ be a sequence of positive numbers, $\varepsilon_n < \frac{1}{2}$ for every n (in the opposite case I take instead of ε_n the $\min(\varepsilon_n, \frac{1}{3})$). We construct, according to Lemma 4, a sequence of numbers > 0, $\{\varepsilon_n'\}$, $\varepsilon_n' \leq \varepsilon_n$ and we choose a sequence of positive numbers $\{n_k\}$, $n_k \to +\infty$, and a sequence of polynomials $P_k(z)$ in the following manner: we choose an arbitrary point $z_0 \in E$ and lay down:

$$P_1(z) = (z-z_0)^2, \quad n_1 = 2. \tag{23}$$

According to Lemma 3 there exists for the number ε_3' a point $z_0' = z_{01} \in E$ and a polynome $P_2(z)$, such that:

$$|z_{00}-z_{01}| < \varepsilon_3', \quad (z_{00} = z_0),$$

$$|P_1(z)-P_2(z)| < \varepsilon_3' \quad \text{for} \quad |z| \leq 1.$$

and $P_2(z)$ is of degree $n_2 > 2$ and has in z_{00}, z_{01} zeros of degree ≥ 3. Suppose the polynomial $P_k(z)$ of degree n_k is already defined, which has in the points $z_{0\alpha_1\alpha_2\ldots\alpha_{k-1}} \in E$ zeros of degree $\geq k+1 \geq 2$, where $\alpha_i = 0$ or 1, and the sequence of indices passes all possible combinations of 0 and 1, $z_{0\alpha_1\alpha_2\alpha_3\ldots\alpha_p 0} = z_{0\alpha_1\alpha_2\alpha_3\ldots\alpha_p}$ (*i.e.*, we can in an arbitrary number add zeros to the sequence of indices), and $z_{0\alpha_1\alpha_2\ldots\alpha_p} = z_{0\beta_1\beta_2\ldots\beta_r}$ if and only if, (after having added zeros to the shorter sequence), when $p=r$, $\alpha_1=\beta_1$, $\alpha_2=\beta_2, \ldots \alpha_p=\beta_p$, $|z_{0\alpha_1\alpha_2\ldots\alpha_p 1}-z_{0\alpha_1\alpha_2\ldots\alpha_p 0}| < \varepsilon'_{p+3}$ for $p = 0, 1, 2, \ldots, k-2$. According to Lemma 3 there exists a polynomial $P_{k+1}(z)$ of degree $n_{k+1} > n_k$, by $\min(\varepsilon'_{k+2}, \varepsilon'_{n_k+1})$ instead of ε, and:

$$|z_{0\alpha_1\alpha_2\ldots\alpha_{k-1}0} - z_{0\alpha_1\alpha_2\ldots\alpha_{k-1}1}| < \varepsilon'_{k+2}$$

$$P_{k+1}^{(m)}(z) = 0 \quad \text{for} \quad m = 0, 1, 2, \ldots, k+1, \quad z = z_{0\alpha_1\alpha_2\ldots\alpha_k},$$

$$|P_{k+1}^{(m)}(z)-P_k^{(m)}(z)| < \varepsilon'_{n_k+1} \quad \text{for} \quad |z| \leq 1, \quad m = 0, 1, 2, \ldots, k-1. \tag{24}$$

In that way all the polynomials $P_k(z)$ are defined by induction. The sequence $P_k(z)$ converges uniformly in $|z| \leq 1$ with all derivatives. We have, indeed: for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for $k \geq N$, $\varepsilon'_{n_k} < \varepsilon$. Taking $N_1 = \max(n+1, N)$, we have from (24).

$$|P_k^{(n)}(z) - P_{k+l}^{(n)}(z)| < \varepsilon'_{n_k+1} + \varepsilon'_{n_{k+1}+1} + \cdots + \varepsilon'_{n_{k+l-1}+1} < \sum_{i=n_k+1}^{\infty} \varepsilon'_i \leq \varepsilon'_{n_k} \text{ for } k > n, \tag{25}$$

$|z| \leq 1$, *i.e.*. in particular for $k \geq N_1$, $|P_k^{(n)}(z)-P_{k+l}^{(n)}(z)| < \varepsilon$ in $|z| \leq 1$, hence $\{P_k^{(n)}(z)\}$ fulfills in $|z| \leq 1$ uniformly Cauchy's condition. Denoting $f^{(n)}(z) = \lim\limits_{k\to\infty} P_k^{(n)}(z)$ we obtain from (25) with $l \to \infty$:

$$|P_k^{(n)}(z)-f^{(n)}(z)| \leq \varepsilon'_{n_k} \quad \text{for} \quad k > n, \quad |z| \leq 1. \tag{26}$$

The sequence ε'_n being a decreasing one, we obtain $\varepsilon'_{n_k} \leq \varepsilon'_{n_k-n} \leq \varepsilon_{n_k-n}$; and the degree of $P_k^{(n)}(z)$ being n_k-n, it follows from $|P_k^{(n)}(z)-f^{(n)}(z)|\leq \varepsilon_{n_k-n}$ for $k>n$, $|z|\leq 1$, that *all the functions* $f^{(n)}(z)$, $n=0,1,2,\ldots$ *are quasi-analytic* $(B\varepsilon_n)$ *in* $|z|\leq 1$. In value of the uniform convergence, $f^{(n)}(z)$ is the n-th derivative of $f(z)$ in $|z|\leq 1$. From (23), (26) and from $\varepsilon'_{n_1}=\varepsilon'_2\leq\varepsilon_2<\frac{1}{2}$ it follows, that

$$|P_1(0)-f(0)|<\tfrac{1}{2},\quad |f(0)|>|z_0^2|-\tfrac{1}{2}=1-\tfrac{1}{2}=\tfrac{1}{2}>0,$$

and therefore $f(z)$ *does not vanish identically*. In every point $z_{0\alpha_1\alpha_2\ldots\alpha_k}$ the derivative of the range $k+1$ or lower of the polynomial $P_{k+1}(z)$ vanishes, and so do the derivatives of the polynomial $P_{n+1}(z)$, $n\geq k$, according to the agreement concerning the adding of zeros to the sequence $\alpha_1\alpha_2\ldots\alpha_k$. Passing to the limit $(n\to\infty)$, we obtain

$$f^{(m)}(z)=0\quad\text{for}\quad z=z_{0\alpha_1\alpha_2\ldots\alpha_k},\quad m\leq k+1,$$

but adding zeros to the sequence $\alpha_1\alpha_2\ldots\alpha_k$ we can get free of the condition $m\leq k+1$. The $f^{(m)}(z)$ being continuous in $|z|\leq 1$, the set of their zeros is closed; denoting by $E_1\subset E$ the fermeture of the set of points $z_{0\alpha_1\alpha_2\ldots\alpha_k}$ for all finite sequences of 0 and 1 $(=\alpha_i)$, $k=1,2,3,\ldots$, we obtain

$$f(z)=f^{(m)}(z)=0\quad \textit{for every}\ z\in E_1,\quad m=1,2,3,\ldots$$

From the construction of $z_{0\alpha_1\alpha_2\ldots\alpha_k}$ it follows, that this set is dense in itself, for $z_{0\alpha_1\alpha_2\ldots\alpha_k}$ is a point of condensation of the sequence $z_{0\alpha_1\alpha_2\ldots\alpha_k1}$, $z_{0\alpha_1\alpha_2\ldots\alpha_k01}$, $z_{0\alpha_1\alpha_2\alpha_3\ldots\alpha_k001}$ etc., whose distances from $z_{0\alpha_1\alpha_2\ldots\alpha_k}$ are resp. $<\varepsilon'_{k+3}$, ε'_{k+4}, ε'_{k+5}, etc., and $\lim\limits_{n\to\infty}\varepsilon'_n=0$. It follows, that E_1 is *perfect*, and, as *not empty*, *of continuum-power*; hence $f(z)$ fulfills the conditions of the theorem.

The sequence ε_n being arbitrary we can obtain functions approximated by polynomials with any degree of precision, *e.g.*, such that $|P_k(z)-f(z)|\leq\delta^{n_k^{n_k}}$, $0<\delta<1$ and not only $\leq M\delta^{n_k}$ (after M. Bernstein). On the second side, the set of functions quasi-analytic $(B\varepsilon_n)$ in Z for all sequences $\{\varepsilon_n\}$, $\varepsilon_n\to 0$ is the set of all functions, that can be in Z uniformly approximated by polynomials, and therefore, if Z is the interval $[0,1]$, is the set of all continuous functions on this interval. It follows that in $[0,1]$ *we can, for certain sequences* $\{\varepsilon_n\}$, *get free of the condition*: *transfinite diameter of the set of zeros* $=0$, and even of the condition: the measure $=0$. On the contrary, when Z is the circle $|z|\leq 1$, we *cannot*, according to the theorem of MM. Riesz, obtain, for any sequence $\{\varepsilon_n\}$, a function, the set of zeros of which would be *of measure* >0, for $f(z)$ is then analytic in $|z|<1$, bounded in $|z|\leq 1$. I ask, *if we can, for certain sequences* $\{\varepsilon_n\}$, *obtain a function quasi-analytic* $(B\varepsilon_n)$ *in* $|z|\leq 1$, *the set of zeros of which has the transfinite diameter* >0, and, if, *for every sequence* $\{\varepsilon_n\}$ *and every set* E *of points of the segment* $[0,1]$ *or the perimeter of* $|z|\leq 1$ *of transfinite diameter* 0 (*in particular for* $\varepsilon_n=M\delta^n$, $0<\delta<1$), *there exists a function quasi-analytic* $(B\varepsilon_n)$ *in* $[0,1]$ *or in* $|z|\leq 1$, *not vanishing identically and such that* $f(z)=0$ (*and ev.* $f^{(n)}(z)=0$) *for every* $z\in E_1\supset E$. It is not impossible, that the set of zeros of such a function, especially for a sequence ε_n, that converges to 0 stronger than δ^n, must fulfill a condition stronger than the condition: transfinite diameter = 0.

WARSZAWA,
POLAND.

References

Mazurkiewicz, St. and Szmuszkowiczowna, H., (1936). *Bull. de l'Acad. Pol. des Sc. et des Lettr.*
Mazurkiewicz, St. and Szmuszkowiczowna, H., (1937). *Bull. de l'Acad. Pol. des Sc. et des Lettr.*

GENERATION OF MESONS AND ITS DEPENDENCE ON MESON SPIN

By

S. K. Chakrabarty

(*Received December 20, 1947*)

Abstract. Mesons obtained at sea-level are produced within the atmosphere but doubts exist regarding the primary which produces these mesons and also the process through which they are generated. Several hypothesis have been postulated of which the theory for the production of mesons by proton-nucleon collisions seems to be more reasonable. On this theory a proton by colliding with a nucleon will generate mesons as well as recoil nucleons and will also loose energy by ionization. In the present paper approximate expressions for the cross sections for different processes have been obtained both for a *transverse* as well as a *pseudoscalar* meson. These have later been used for the calculations of (i) the energy spectrum of nucleons at different levels of the atmosphere, (ii) energy spectrum of mesons (both transverse and pseudo-scalar separately) at different levels of the atmosphere produced by primray protons of different energies and (iii) the integral spectrum of mesons at the different levels for an assumed energy spectrum of primary protons incident at the top of the atmosphere. The energy spectrum of nucleons obtained here differs considerably from similar results obtained by Peng and the reasons for such differences have been explained. The energy spectrum of the mesons, whether transverse or pseudo-scalar, produced by a primary proton are nearly similar except for the fact that the number of mesons in different energy regions are larger by a factor 2-3, if the mesons are transverse than if they are pseudoscalar. But this fact cannot be used to ascertain the spin of sea level mesons unless the intensity as well as the energy spectrum of the primary protons are accurately known The energy spectrum, however, depends critically on the proper life time of the meson at rest and comparison of the results of the present paper with the observed results of Wilson show that $\tau \sim 2.7 \times 10^{-6}$ sec. The observed knee of the latitude effect can however, be explained whether the mesons are transverse or pseudo-scalar.

The theory of meson production by proton-nucleon collisions have been discussed by several authors (Hamilton, Heitler and Peng, 1943). In view of the difficulties in solving the exact equations which, however, is based on the divergence free quantum theory of damping they have used the simplified procedure of Weizsacker-Williams. In deriving the cross-sections for the different processes, the nucleon-nucleon collisions have been divided into two groups. When a fast particle having an energy E collides with a particle at rest and thereby loses energy ϵ (in the form of virtual quanta) of which the quantum or meson emitted takes up an energy ϵ' and the energy transferred to the particle at rest is $\epsilon - \epsilon'$. The two cases considered are

$$\text{(i)} \quad E \gg \epsilon, \quad \text{and} \quad \text{(ii)} \quad E \sim \epsilon, \text{ but } \epsilon - \epsilon' \ll M. \qquad (1)$$

Hamilton, Heitler and Peng deduced the results of the case (ii) by extrapolating the results obtained in (i). Heitler (1945) has however shown that this is not possible, but that the results of case (ii) can be derived from that of (i) by applying a Lorentz transformation. It may be mentioned here that when $m \ll M$, where m and M are the masses of the fast particle and the particle at rest respectively; the contribution of the case (ii) only is important (*cf.* bremsstrahlung). It is however, clear that the restrictions imposed by (1) do not cover the whole range of values of ϵ and ϵ' with which we are concerned, but we should get *fairly* accurate expressions for the cross-sections if we take for (1(i)) $\epsilon < E/2$. The inaccuracy involved in this assumption is however, inherent, in this method of treatment but we expect that it will not seriously affect the results derived in this paper. Heitler has however, suggested that we may expect errors of the order of magnitude of a factor 2 for the energy spectrum or total number of mesons.

Let $\phi(E, \epsilon)d\epsilon$ be the cross-section for the production of a meson (ϕ_{ps} or ϕ_{tr} according as the meson is a pseudoscalar or transverse) having energy between ϵ and $\epsilon + d\epsilon$, by a fast nucleon of energy E. The cross-sections have been calculated by Heitler (1945). There is no conclusive evidence either way as to whether the masses of the two kinds of mesons are equal or not but following Heitler we assume that they are equal and also equal to 1/10 of the proton mass. It may, however, be mentioned that in order to explain the observed quadrupole moment of the Deuteron, it is necessary to assume different masses for the different types of meson (*cf.* also the effect on burst production as pointed out by us (Chakrabarty & Majumdar, 1944)) but we shall not take this into account here and adopt the expressions for the cross sections derived by Heitler (1945). Using the natural meson units for which $c = \hbar = \mu = 1$, cross-sections are in units of $(\hbar/\mu c)^2 = 4.8 \times 10^{-26}$ and energies are expressed in units of $\mu c^2 = 0{\cdot}94 \times 10^8$.*e.v.*, we have

$$\begin{aligned}\phi_{ps}(E, \epsilon) = \Big[&\tfrac{4}{3}f^2(D_t + D_p)\frac{1}{\epsilon^3}\left(1 - \frac{\epsilon}{M}\right)H\left(\epsilon - \frac{1}{f}\right)H(\epsilon - \tfrac{1}{2}M)\\ &+ \tfrac{2}{3}Mf^2(D_t + D_p)\frac{E - \epsilon}{E^2}\left\{\left(f^3 - \frac{8}{M^3}\right)H(\epsilon - 1)H(\tfrac{1}{2}E - \epsilon)\right.\\ &\qquad\left.+ \frac{8}{M^3}\left(\frac{E^3}{\epsilon^3} - 1\right)H\left(\epsilon - \frac{2E}{Mf}\right)H(E - \epsilon)\right\}\Big]\\ &+ \left[\frac{64}{5}\frac{f^2}{M}D'.\frac{1}{E^{1/2}}.\left(\frac{1}{\epsilon^{3/2}} - (2/E)^{5/2}.\epsilon\right)H(\epsilon - \tfrac{1}{2}M)H(\tfrac{1}{2}E - \epsilon)\right.\\ &\qquad\left.+ \frac{64\sqrt{2}}{M}f^2D'\frac{1}{E^3}(E - \epsilon)\left\{\left(\frac{E - \epsilon}{M}\right)^{\frac{1}{2}} - 1\right\}H(\epsilon - 1)H(E - M - \epsilon)\right]\\ = \phi_{psNR} + \phi_{psBR}&\end{aligned} \tag{2a}$$

$$\begin{aligned}\phi_{tr}(E, c) = 2\phi_{psNR} + &\left[\frac{16}{3}.\frac{f^2}{M}D_t\left(\frac{1}{\epsilon^2} - \frac{8}{E^3}\epsilon\right)H(\epsilon - \tfrac{1}{2}M)H(\tfrac{1}{2}E - \epsilon)\right.\\ &\left.+ \frac{32f^2}{M}D_t\frac{1}{E^3}(E - \epsilon)\left(\frac{E - \epsilon}{M} - 1\right)H(\epsilon - 1)H(E - M - \epsilon)\right]\\ = \phi_{trNR} + \phi_{trBR}&\end{aligned} \tag{2b}$$

where $H(x)$ is the Heavisides function and is defined by the relation

$$\begin{aligned}H(x) = \int_{-x}^{\infty}\delta(x')dx' &= 1 \quad \text{if} \quad x > 0\\ &= 0 \quad \text{if} \quad x < 0\end{aligned}$$

and

$$D_t = 165, \quad D_t + D_p = 200, \quad D_t + 2D_p = 230, \quad D' = 50, \quad f^2 = 0{\cdot}13. \tag{3}$$

The above expressions are however, restricted to be valid in the region for which $E \gg M$; but they can also be used even if E is not much larger than M, provided E represents the total energy and not the kinetic energy. Let $\psi(E, E')dE'$, be the cross-section for the production of a recoil nucleon of energy between E' and $E'+dE'$, by a nucleon of energy E (E' is the $K.E.$ of the recoil nucleon). Since recoil nucleons are produced only through the process I, we have

$$\psi(E, E') = \int_{E'+1}^{E/2} \Phi(\epsilon, \epsilon - E')dt \tag{4}$$

where $\Phi(\epsilon, \epsilon')d\epsilon$ is the differential cross sections for the production of a pseudoscalar or transverse meson of energy ϵ' by a vertual meson ϵ, and have been given by Heitler. Taking into consideration the fact that recoil nucleons are also produced together with the production of neutral mesons, we have after simplifications

$$\psi(E, E') \approx H(\tfrac{1}{2}E - E' - \tfrac{1}{2}M) \Big[\frac{4f^2}{M}(D_t + 2D_p)\Big\{\frac{1}{(E'+\frac{1}{2}M)^2} + \frac{M}{(E'+\frac{1}{2}M)^3}$$
$$- \frac{12}{E^3}\Big(1 - \frac{4}{3}\cdot\frac{E'}{E}\Big)\Big\} + 3Mf^2(D_t + D_p)(1 - 2/M)\frac{1}{E'^3} + O\Big(\frac{1}{E'^4}\Big) + \cdots\Big] \tag{5a}$$

If however, $\frac{1}{2}E < E' + \frac{1}{2}M$; $3M/2 < E' + \frac{1}{2}M$, then the contribution to $\psi(E, E')$ comes through the N. R. formula alone and in that case

$$\psi(E, E') \approx 3Mf^2(D_t + D_p)\Big(1 - \frac{2}{E - 2E'}\Big)\frac{1}{E'^3} + O\Big(\frac{1}{E'^4}\Big) \tag{5b}$$

Let $\chi(E, E')dE'$ be the cross section for the fast nucleon of energy E to lose and energy between E' and $E'+dE'$ through the production of reeoil nucleons and mesons. The fast nucleon loses energy through both the processes. Hence denoting the contributions to $\chi(E, E')$ through the two processes by χ^I and χ^{II}, respectively and taking into account the effect of neutral mesons we have

$$\chi^I(E, t) = \frac{3}{2}\int_{\epsilon'=1/f}^{\epsilon} \Phi(\epsilon, \epsilon')dt'$$
$$= \Big\{3Mf^2(D_t + D_p)\frac{1}{\epsilon^3}\Big(f - \frac{1}{\epsilon}\Big)H\Big(\epsilon - \frac{1}{f}\Big)H(\tfrac{1}{2}M - \epsilon)\Big\} + \Big\{3Mf^2(D_t + D_p)\frac{1}{\epsilon^3}\Big(f - \frac{2}{M}\Big)$$
$$+ \frac{24}{M}f^2(D_t + 2D_p)\frac{1}{\epsilon^2}\Big(\frac{1}{2} - \frac{M^2}{8\epsilon^2}\Big)\Big\}H(\epsilon - \tfrac{1}{2}M), \tag{6a}$$

and

$$\chi^{II}(E, \epsilon) = \frac{48f^2}{M}(D_t + 2D_p)\frac{(E-\epsilon)^2}{E^3}\Big(1 - \frac{M}{E-\epsilon}\Big) + Mf^2(D_t + D_p)\Big(f^3 - \frac{8}{M^3}\Big)\frac{E-\epsilon}{E^2}H\Big(\frac{2E}{Mf} - \epsilon\Big)$$
$$+ Mf^2(D_t + D_p)\frac{8}{M^3}\Big(\frac{E^3}{\epsilon^3} - 1\Big)\frac{E-\epsilon}{E^2}H\Big(\epsilon - \frac{2E}{Mf}\Big), \quad \text{if} \quad 1 \lesssim \epsilon \lesssim E - M. \tag{6b}$$

It may be pointed out that the lower limit for the E. R. formula has been taken, following Heitler, as $M/2$ but this is not strictly correct. It is evident that

$$\int \epsilon\chi(E,\, \epsilon)d\epsilon = E\phi_{\text{en.loss}} = E\phi^{I}_{\text{en loss}} + E\phi^{II}_{\text{en loss}}$$

where

$$E\phi^{I}_{\text{en.loss}} \approx \frac{12f^2}{M}(D_t+2D_p)\log\frac{E}{M} - \frac{6f^2}{M}(D_t+2D_p) + \tfrac{3}{2}Mf^2(D_t+D_p)\left(f^2-\frac{8}{M^2}\right) + O(1/E)$$

$$= 35.88 \log (E/M) + 7.02 - 5.46 \quad \text{(7a)}$$

and

$$E\phi^{II}_{\text{en.loss}} \approx E\left[\frac{48f^2}{M}(D_t+2D_v)\cdot\frac{1}{12M} + Mf^2(D_t+D_p)\left\{\left(f^1-\frac{8}{M^3}\right)\left(\frac{2}{M^2f^2}-\frac{8}{3M^3f^1}\right)\right.\right.$$

$$\left.\left. + \frac{8}{M^3}\left(\frac{Mf}{2} - 1 + \log\frac{2}{Mf} - \frac{1}{6} - \frac{8}{3M^3f^3} + \frac{2}{M^2f^2}\right)\right\}\right]$$

$$+ \frac{48f^2}{M}(D_t+2D_p)\left(-\frac{1}{6}-\frac{8}{M^2}\right) + O(1/E)$$

$$= 2{\cdot}478E - 28{\cdot}92 - 11{\cdot}48 \quad \text{(7b)}$$

Heitler (1945) has obtained similar expressions without the last terms in (7a) and (7b) which are the contributions of the N. R. terms. It is evident that the energy loss due to process I corresponds to the ionization loss in the cascade showers of soft particles and that due to process II corresponds to radiation loss. In the following we shall treat the two parts of $\chi(E,\, \epsilon)$ separately and take $\chi(E,\, E') = \chi^{II}(E,\, E')$ and treat $\chi^{I}(E,\, E')$ as a constant energy loss and combine it with the ionization loss.

Let us choose the unit of length in such a way that the number of nucleons contained in a cylinder of the material of unit length and of unit cross section (*i.e.*, $(\hbar/\mu c)^2$) is unity. We measure the lengths in this unit by t, so that, if l cm. corresponds to a unit length in the unit defined above we have,

$$l \times (\hbar/\mu c)^2 NA = 1,$$

where N is the number of atoms or molecules per c.c. and A the atomic or molecular weight of the material so that NA is the number of nucleons per c.c. of the material. So that the values of l for different materials are given by

	air	H_2O	Pb
l in cms.	34200	38.39	8.54

This is practically the length through which a fast particle has to travel in order to lose an energy (nearly) equal to 10^8 *e.v.* The unit of length chosen by Heitler is that length through which a fast particle has to travel in order to lose an energy equal to μc^2 by ionization. The unit of length chosen in this paper are the same as that taken by Heitler particularly for air with which we are interested. In air, however, this unit coincides with the radiation unit defined earlier (Bhabha & Chakrabarty, 1943) for the cascade shower of soft particles.

Consider a layer of substance on the surface of which primary protons with an energy distribution $F(E_0)dE_0$ impinge normally. We wish to find out the energy distribution of the mesons and the nucleons at any depth below the surface of the layer. The mesons together with recoil nucleons are produced by the fast nucleons and they are absorbed through ionization and β-decay. Let $P(E, t)dE$ be the total number of nucleons at depth t and having energies between E and $E+dE$ and $M(E, t)$ the total number of mesons having energies between E and $E+dE$. Considering the production and absorption through the different processes discussed above it is evident that $P(E, t)$ and $M(E, t)$ must satisfy the following equations:

$$\frac{\partial P(E,t)}{\partial t} - \beta\frac{\partial P(E,t)}{\partial E} = \int_E^\infty P(E',t).\chi(E',E'-E)dE'$$

$$-\int_0^E P(E',t).\chi(E,E')dE' + \int_E^\infty P(E',t).\psi(E',E)dE' \quad (9a)$$

$$\frac{\partial M(E,t)}{\partial t} - \gamma\frac{\partial M(E,t)}{\partial E} = \int_E^\infty P(E',t).\phi(E',E)dE' - \frac{b}{Et}M(E,t) \quad (9b)$$

where β, γ are the rates of ionization loss of the nucleons and mesons, (β including χ^I) and b depends on the proper life time of the meson at rest and for the atmosphere,

$$b = \frac{1}{c\tau}lt_0 \approx 9{\cdot}5 \quad \text{if} \quad \tau = 2{\cdot}7\times10^{-6}\ \text{sec.},$$

where t_0 is the value of t corresponding to sea level measured from the top of the atmosphere. This value of τ has been taken from a measured ratio of τ/μ made by Rossi and Hall (1941) and taking $\mu = 184m$. The value of b thus determined is however, not very accurate and may have an error to the extent of 20%. We shall have to find the solution of the above equations subject to the proper boundary condition, *viz.*, at $t = 0$

$$M(E,t) = 0 \quad \text{and} \quad P(E,t) = \delta(E_0 - E),$$

if the primary is a proton of energy E_0.

Following the methods applied in the case of cascade showers of soft particles and applying Mellin's transformation, where

$$p(s,t) = \int_0^\infty E^{s-1}P(E,t)dE$$

we have after some simplifications from (9a) and neglecting small quantities of higher order,

$$\frac{dp(s,\ t)}{dt}+A_s p(s,\ t)+B_s p(s-1,\ t)=0 \tag{10}$$

where

$$A_s = a_0\left(\frac{1}{3}-\frac{1}{s+2}\right)+b_0\left(\frac{1}{2}-\frac{1}{s+1}\right)$$

$$B_s = \beta(s-1)-a_0\left(\frac{1}{2}-\frac{1}{s+1}\right)-\frac{a_0}{3\times 2^s}\left(\frac{1}{s-2}-\frac{3}{s}+\frac{2}{s+1}\right)$$

$$a_0 = \frac{48f^2}{M}(D_t+2D_p)\approx 143{\cdot}5, \quad b_0 = Mf^2(D_t+D_p)\left(f^3-\frac{8}{M^3}\right)\approx 10{\cdot}06$$

$$\beta \approx \frac{12f^2}{M}(D_t+2D_p)\log\frac{E}{M}+17 = 36\log\frac{E}{M}+17 = 36\log{\cdot}16E \approx 50 \tag{10a}$$

It can be easily shown that an obvious solution of (10) is given by

$$p(s,\ t)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}e^{rt}dr.\frac{E_0^{\,s-1}}{r+A_s}\left[1+\sum_{n=1}^{\infty}\left(-\frac{1}{E_0}\right)^n\prod_{i=1}^{n}\frac{B_{s+1-i}}{r+A_{s-i}}\right] \tag{11}$$

and also as in the case of electron cascade

$$P(E,\ t)\approx P_0(E,\ t)=\frac{1}{2\pi iE_0}\cdot\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E+g(s,\ t)}\right)^s\exp\left(-A_s t\right).ds \tag{12}$$

where

$$g(s,\ t)=\frac{B_{s+1}}{s(A_{s+1}-A_s)}\cdot\left\{1-\exp\left(-\overline{A_{s+1}-A_s}t\right)\right\} \tag{13}$$

Peng (1944) obtained an expression for $P(E,\ t)$ without taking into consideration the large energy loss arising through the process (ii) and as such the multiplication in his cascade process was very much over-emphasised. It may be pointed out here that since the cross section for the production of recoil nucleon is of the order of $1/E^2$, whereas that for the energy loss by process (ii) is of the order of $1/E$, the multiplication of the nucleons is very small as compared to the absorption. Moreover the energy loss through process (i) reduced to a certain extent the effect of multiplication through the production of recoil nucleons since the cross sections for the two processes depend in a similar way on the energy of nucleon.

If now $N(E,\ t)$ be the total number of nucleons at depth t and having energies equal to or greater than E, produced by a primary nucleon of energy E_0, then

$$N(E,\ t)=\int_{E}^{\infty}P(E,\ t)dE=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E+g(s,\ t)}\right)^{s-1}\cdot\frac{1}{s-1}\exp\left(-A_s t\right).ds. \tag{14}$$

The above integral has been evaluated by the saddle point method for some values of t and E_0 and the values thus obtained are given in Table I.

TABLE I

Values of $N(E, t)$, for $E = 10$

t \ $\log_e E_0$	6	8	10	12	15
0.5	0.93	1.07	1.13	1.15	1.17
1.0	0.25 (0.46)	0.81 (0.78)	1.02 (0.91)	1.07 (0.99)	1.10 (1.00)
2.0	...	...	0.40	0.74	0.96

It shows as expected that there is practically no multiplication. The results obtained by Peng differ widely from those obtained above. For small value of t say $t \leq 1$, they are larger by a factor 3 but for large thickness the values obtained by Peng is sometimes more than 100 times larger. The reasons for such a difference is the total neglect of the energy loss which comes through process (ii) which gives A_s in (14). If however, we put $A_s = 0$ in (14) we get values similar to that obtained by Peng. On the contrary if we neglect $g(s, t)$ which means that we ignore the producfion of the recoil nucleon as well as the energy loss produced through process I we get

$$P(E, t) = \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} (E_0/E)^s \exp(-A_s t) ds,$$

$$N(E, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (E_0/E)^{s-1} . \frac{1}{s-1} \exp(-A_s t) ds. \tag{15}$$

The values of $N(E, t)$ for $t = 1$, derived from (15) have also been included in parenthesis in Table 1. It shows that (15) is a good approximation for (14) and the second order effects can be ignored without introducing appreciable error in the final result. Heitler has, however, made a similar approximation in deriving the meson spectrum.

If however we assume that the primary spectrum of the nucleons, *i.e.*, $F(E_0)$ at $t = 0$ is given by

$$F(E_0) = A/E_0^{\alpha+1} \tag{16}$$

then the total number of nucleons at any depth t due to this incident primary and having energies between E and $E + dE$ is given by

$$\bar{P}(E, t) = \int_E^\infty P(E, t) F(E_0) dE_0 = \frac{A}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{E^{\alpha+1}} \frac{\exp(-A_s t)}{s - \alpha - 1} ds = \frac{\exp(-A_{\alpha+1} t)}{E^{\alpha+1}}. \tag{17}$$

Assuming, $\alpha = 1·5$, we get $A_{\alpha+1} = 3·75$, so that

$$\bar{P}(E, t) = A . \frac{e^{-3·75t}}{E^{\alpha+1}}. \tag{17a}$$

Heitler and Walsh (1945) have obtained from a consideration of the range-energy relation (*cf.*, eq. 18)

$$\bar{P}(E,\, t) = A.\, \frac{e^{-4\cdot 5t}}{E^{\quad +1}}.$$

It has been pointed out by Hamilton, Heitler and Peng (1943) that the meson production ceases when the energy of the nucleon is equal to $2M$. Such protons will however penetrate a considerable depth further nearly half the atmosphere if the energy loss is purely by ionization loss and will be counted as "penetrating particles." We can therefore assume that for all values of $E \geq 2M$, $P(E,t)$ is given by (12) or (15) but for $M \leq E \leq 2M$ we can take the number of protons as given by $P^*(E,t)$ where

$$P^*(E,\, t) = H(2M-E)P(2M,\, t') \tag{18}$$

and $2M-E = \beta'(t-t')$ and β' is the ionization loss which in the units of the present paper is practically unity.

Let us now calculate the number of mesons $M(E,t)$ at any depth, which satisfies equation (9*b*). An obvious solution is

$$M(E,\, t) = \left(\frac{E}{\gamma t}\right)^{\frac{b}{E+\gamma t}} \int_0^t S(E+\gamma t-\gamma t';\, t')\left(\frac{\gamma t'}{E+\gamma t-\gamma t'}\right)^{\frac{b}{E+\gamma t}} dt' \tag{19}$$

where

$$S(E,\, t) = \int_E^\infty P(E',\, t)\phi(E',\, E)\,dE' \tag{20}$$

and $\phi(E,E')$ is defined by (2). It is clear from the expression derived for $P(E,t)$ that only very small values of t contribute to $S(E,t)$ and consequently for not too small values of $E+\gamma t$ we can neglect $\gamma t'$ in comparison with $E+\gamma t$ without introducing any appreciable error. We then have,

$$M(E,\, t) \approx \left(\frac{E}{\gamma t}\right)^{b/\eta} \int_0^t S(\eta,\, t')\left(\frac{\gamma t'}{\eta}\right)^{b/\eta} dt' \tag{21}$$

where $\eta = E+\gamma t$.

Taking the values of $P(E,t)$ given by (15) we get after simplifications,

$$\begin{aligned} M(E,\, t) &\approx \left(\frac{E}{\gamma t}\right)^{b/\eta} \cdot \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} E_0^{s-1}\,ds \int_0^t \exp(-A_s t')\cdot\left(\frac{\gamma t'}{\eta}\right)^{b/\eta} dt' \int_\eta^\infty E'^{-s}\phi(E',\, \eta)\,dE' \\ &= \left(\frac{E}{\eta t}\right)^{b/\eta} \cdot \frac{1}{2\pi i E_0} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{\eta}\right)^s \frac{\Gamma_{(A_s t)}(1+b/\eta)}{(A_s)^{1+b/\eta}}\chi(s,\, \eta)\,ds \end{aligned} \tag{22}$$

where

$$\chi(s,\, \eta) = \eta^s \int_\eta^\infty E'^{-s}\phi(E',\, \eta)\,dE'$$

and $\Gamma_x(1+p)$ is the incomplete gamma function.

$\chi(s, \eta)$ can now be easily evaluated from (23) by substituting the value of $\phi(E_0, \eta)$ given by (2). So that we get two different expressions for $\chi(s, \eta)$, depending on whether the meson is transverse or pseudoscalar. We then have, *for transverse mesons*

$$\chi(s, \eta) = f^2\Big[\frac{8}{3}(D_t + D_p)\frac{1}{\eta^2}\Big(1 - \frac{\eta}{M}\Big)\frac{1}{s-1}H\Big(\eta - \frac{1}{f}\Big)H\Big(\frac{M}{2} - \eta\Big) + 16\frac{D_t}{M}\cdot\frac{1}{\eta}\,\frac{1}{2^{s-1}}\cdot\frac{1}{(s-1)(s+2)}H\Big(\eta - \frac{M}{2}\Big)$$
$$+\frac{4}{9}M(D_t + D_p)\Big\{\Big(f^3 - \frac{8}{M^3}\Big)\Big(\frac{1}{s} - \frac{k}{s+1}\Big)k^s + \frac{8}{M^3}\Big(\frac{1-k^{s-3}}{s-3} - \frac{1-k^{s-2}}{s-2} - \frac{1}{s(s+1)}\Big)\Big\}$$
$$+\frac{32D_t}{M^2}\Big\{r^s\Big(\frac{1}{s(s+1)} - \frac{r}{(s+1)(s+2)}\Big)\Big]\quad (24a)$$

and *for pseudoscalar mesons.*

$$\chi(s, \eta) = f^2\Big[\frac{4}{3}(D_t + D_p)\frac{1}{\eta^2}\Big(1 - \frac{\eta}{M}\Big)\frac{1}{s-1}H\Big(\eta - \frac{1}{f}\Big)H\Big(\frac{M}{2} - \eta\Big) + \frac{32}{M}D'\frac{1}{2^{s-1/2}}\cdot\frac{1}{(s-\frac{1}{2})(s+2)}\cdot\frac{1}{\eta}H\Big(\eta - \frac{M}{2}\Big)$$
$$+\frac{2}{9}M(D_t + D_p)\Big\{\Big(f^3 - \frac{8}{M^3}\Big)\Big(\frac{1}{s} - \frac{k}{s+1}\Big)k^s + \frac{8}{M^3}\Big(\frac{1-k^{s-3}}{s-3} - \frac{1-k^{s-2}}{s-2} - \frac{1}{s(s+1)}\Big)\Big\}$$
$$+\frac{64\sqrt{2}}{M}D'\Big\{\frac{1}{M^{\frac{1}{2}}\eta^{\frac{1}{2}}}B_r(s+\tfrac{1}{2}, 5/2) - \frac{r^{s+1}}{\eta}\Big(\frac{1}{s+1} - \frac{r}{s+2}\Big)\Big\}\Big]\quad (24b)$$

where $k = \frac{2}{Mf}$, $r = \frac{1}{1+M/\eta}$ and B_r is the incomplete β-function defined by

$$B_r(s+\tfrac{1}{2}, 5/2) = \int_0^r x^{s-1/2}(1-x)^{3/2}dx.$$

The contribution of the different terms in the expression for the cross-sections is evident in equation (24). It appears that the values of $\chi(s, \eta)$ for the different values of s and η is larger by a factor varying between 2 and 3 in the case of transverse mesons as compared to the corresponding values for pseudoscalar mesons. This fact has its effect in the values of $M(E, t)$ which consequently for same E and t is larger in the case of transverse mesons than in the case of pseudoscalar mesons. In order to examine the energy spectrum of the mesons at different depths and produced by primaries of different energies we have evaluated the integral in (22) by the saddle point method The values thus obtained for the case of transverse mesons are given in Table II. The results for the case of pseudoscalar mesons will be similar except for the fact that the intensity will be less roughly by a factor 2.5 and can be easily obtained.

Following Heitler if we assume that the primary spectrum of the protons incident on the top of the atmosphere is given by (16) then we can find in that case the total number of mesons at any depth t and having energies lying between E and $E + dE$ given by $\bar{M}(E, t)$ say, where,

$$\bar{M}(E, t) = \int_{E_0}^{\infty} F(E_0)M(E, t)dE_0$$

TABLE II

Values of $M(E, t) \times 10^3$ for transverse mesons

E_0	t \ E	5	10	15	20	30	40	50
10^2	5	4.6	7.5	6.8	6.0	4.7	3.7	3.0
	15	2.3	4.8	4.4	3.8	2.9	2.4	2.0
	28	1.6	3.0	2.8	2.5	1.2	1.0	0.8
10^3	5	15	20	19	17	15	11	9.8
10^4	5	29	35	31	28	21	17	14
	15	11	16	15	14	13	11	10

where E_θ is the cut-off energy at the latitude of observation θ (say). We then have,

$$\bar{M}(E, t) = A\left(\frac{E}{\eta t}\right)^{b/\eta} \Gamma(1 + b/\eta) . \frac{1}{E_\theta^{\alpha+1}} . \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{E_0}{\eta}\right)^s . \frac{\chi(s, \eta)}{A_s^{1+b/\eta}} . \frac{I(A_s t, b/\eta)}{\alpha+1-s} ds \tag{26a}$$

or

$$\bar{M}(E, t) = A\left(\frac{E}{\eta t}\right)^{b/\eta} \Gamma(1 + b/\eta) . \frac{1}{\eta^{\alpha+1}} . \frac{\chi(\alpha+1, \eta)}{(A_{\alpha+1})^{1+b/\eta}} I(A_{\alpha+1} t ; \; b/\eta) \tag{26b}$$

according as $E_\theta >$ or $< \eta$ where $I(x, p)$, is connected with the gamma function and is defined by the relation

$$I(x, p)\Gamma(1 + p) = \Gamma_x(1 + p).$$

I have derived the values of $\bar{M}(E, t)$ as given in (26b) for different values of t and for both the types of mesons and the results are given in table III.

It shows that though the maximum of the energy distribution lies in the neighbourhood of 10 in both the cases, the intensity at the maximum is larger by a factor lying between 2-3 in the case of transverse mesons. It also shows that although $\bar{M}(E, t)$ is zero at $t = 0$, it continuously increases even up to $t = 2$, as we go up in the atmosphere which shows that the maximim of altitude-intensity curve for the total number of mesons lies at $t < 2$, and that most of mesons are produced within a very thin layer at the top of the atmosphere. The values of $\bar{M}(E, t)$ depends critically on the value of b and to examine its variation with b we have also calculated the values of $\bar{M}(E, t)$ for different values of b, *viz.*, 7.6 and 95.0. The results of observations (Wilson, 1946) show that at sea level the meson spectrum has maximum near about 8.0, which when compared with the results obtained here shows that b must be of the order of 9.0 and cannot be much higher and and consequently $\tau \approx 2.7 \times 10^{-6}$ sec.

To examine the latitude effect I have evaluated the values of $\bar{M}(E, t)$ as given by

TABLE III

Values of $\bar{M}(E, t) \times 10^5/At^2$ for transverse mesons

t \ E	5	10	15	20	30	40	50	60
5	18	16	14	11	7.1	4.8	3.2	2.4
15	4.8	6.2	5.3	4.7	3.5	2.6	2.1	1.8
23	2.9	3.9	3.5	3.2	2.5	1.9	1.7	1.5

Values of $\bar{M}(E, t) \times 10^5/At^2$ for pseudoscalar mesons

t \ E	5	10	15	20	30	40	50	60
5	5.8	6.6	5.7	4.5	2.8	1.8	1.3	0.9
15	2.0	2.7	2.2	1.9	1.4	1.0	0.7	0.6
23	1.0	1.7	1.4	1.2	0.9	0.7	0.6	0.5

(26*a*) for some values of E_θ which shows that the (sea-level) meson intensity increases as we proceed from equator to higher latitudes till about the latitude for which $E_\theta \approx 23$, after which it is constant and thus explains the observed "knee" of the latitude effect.

The results obtaind in the present paper show that the observational results relating to the meson energy spectrum and the latitude effect can be explained by both types of mesons. The altitude-intensity curve, however, will be different in magnitude though not appreciably in shape but this cannot be used unless the total number of protons incident at tha top of the atmosphere as well as their energy spectrum is known very accurately. We are therefore not in a position, with the help of these measurements to decide whether the sea-level mesons are pseudoscalar or transverse.

COLABA OBSERVATORY,
BOMBAY.

References

Bhabha and Chakrabarty, (1943), *Proc Roy. Soc.* (A), **181**, 267.
Chakrabarty and Majumdar, (1944), *Phys. Rev.*, **65**, 206.
Hamilton, Heitler and Peng, (1943), *Phys. Rev.*, **64**, 78
Heitler and Walsh, (1945), *Rev. Mod Phys*, **17**, 252
Heitler, (1946), *Proc. Roy. Irish. Acad.*, **50**, 155.
Heitler and Peng, (1943), *Proc. Roy. Irish. Acad.*, **49**, 101.
Peng, (1944), *Proc. Roy. Irish. Acad.*, **49**, 245.
Rossi and Hall, (1941), *Phys. Rev.*, **59**, 223.
Wilson, (1946), *Nature*, **158**, 414